Foster Care: A Dynamic Matching Approach *

Diana E. MacDonald†

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Abstract

This paper studies the two-sided dynamic matching problem that occurs in the US foster care system. In this market, foster parents and foster children can form reversible matches, which may separate, continue in their reversible state, or transition to permanency via adoption. I first present an empirical analysis that yields new stylized facts on the match transition of children in foster care. Thereafter, I develop a two-sided search and matching model used to rationalize the empirical facts and carry out model predictions. Interestingly, I find that the presence of a financial penalty on adoption exacerbates the intrinsic disadvantage (being less preferred by foster parents) faced by children with a disability, and it also creates incentives for high-quality matches to not transit to adoption.

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†Economics Research Division, Banco de México, diana.macdonald@banxico.org.mx

1 Introduction

Each year more than a half-million children spend at least one day in the US foster care system, a federal program that costs taxpayers almost US$30 billion dollars annually. The foster care system provides out-of-home care for children removed from their homes due to abuse, maltreatment, neglect, or other reasons.[1] While in foster care, children are placed in institutional care or foster family homes (commonly known as *foster parents*),[2] and can experience placement separation when move from one foster home to another, or from a foster home to institutional care. The stay in foster care is meant to be temporary until children can reunite with their birth families, but when reunification is not possible, children are relinquished for adoption.[3] Each year, close to 18% of children in foster care are at risk of experiencing long-term care if they are not adopted. In fact, more than $20,000$ children leave foster care each year without an adoptive family, and out of those children, less than 3% will earn a college degree, and almost 20% will become homeless.[4] Even though adoption is a preferred alternative to long-term care,[5] parents in the foster care system face a financial penalty on adoption since the monthly payments they received (from the state child welfare agency) are lower as an adoptive parent than as a foster parent, and often fall to zero. Moreover, parents are responsible for the medical and educational expenditures of adopted children, while for children being fostered, these expenses are covered by the state child welfare agency.

This paper studies both, theoretically and empirically, the two-sided dynamic matching problem that occurs in the US foster care system. In this market, foster parents and foster children can form reversible matches, which may separate, continue in their reversible state, or transition to permanency via adoption. I start by presenting an empirical analysis that yields new stylized facts related to match

[1] A child can enter foster care for several reasons such as sexual or physical abuse, parents' drug or alcohol addictions, parents' incarceration, parents' inability to provide care, parents' death, inadequate housing, abandonment, child's behavioral problem, or child's addiction.

[2] Foster homes are private homes licensed to provide 24-hour care for children in a family-based environment. Institutional care are licensed facilities that provide 24-hour care for several children at once (groups from seven to twenty), and it includes group homes, shelter care, and other institutions.

[3] By federal law, if a child has been in foster care for at least 15 of the last 22 months, the process to terminate her parental rights must be started immediately. Further, a judge can decide to terminate parental rights at any moment in time if it is in the best interest of the child.

[4] Source: National Foster Youth Institute.

[5] Adoption is a better alternative to long-term care for two main reasons. First, maintaining a child in long-term care is more expensive than adoption (Barth, 1993; Barth et al., 2006; Hansen, 2008). Second, adoption generates better outcomes for children. Triseliotis (2002) and Hansen (2008) show that children who are adopted exhibit better social and educational outcomes.

transitions of children in foster care and their exit through adoption. Thereafter, I develop a two-sided search and matching model where children are heterogeneous in their disability status, children search for parents while matched to another parent, matches differ in their quality, and parents receive a smaller payoff when adopting than fostering (capturing the financial penalty on adoption). The theoretical model will allows us to understand why certain children are more likely to have their matches separated and why certain children are less likely to be adopted. In addition, the penalty on adoption might have a different effect on certain children, and it might influence the type of matches that transit to adoption (in terms of match quality). The main insight of the theoretical model is that parents face the following trade-off when deciding to adopt: receive the adoption penalty in exchange for eliminating the likelihood that the child separates the match in the future.[6] Hence, match separations play a crucial role in adoption by influencing the incentives of foster parents to adopt. My main result is that the penalty on adoption exacerbates the disadvantage faced by children with a disability (being less preferred by parents), and it also creates incentives for high-quality matches to not transit from a reversible fostering to adoption.

Using a rich panel dataset, describing the universe of children relinquished for adoption in the US foster care system over the period 2010 to 2016, I first document that the presence of a disability: (**Fact 1**) *decreases* the probability that a child transits to permanency via adoption, (**Fact 2**) *increases* the probability that a foster placement separates, (**Fact 3**) *decreases* the probability that a child transits from institutional care to a foster home (becomes foster matched), and (**Fact 4**) *increases* the probability that a child transits from a foster home to institutional care (becomes unmatched). I focus my analysis on disability for two reasons.[7] First, most of the efforts made to increase adoption target children with a disability. Second, the adoption penalty might be higher for children with a disability as parents are responsible for higher medical expenditures. It is important to highlight that with the data available for the analysis is not possible to make any statement regarding what type of matches, in terms of match quality, are more likely to form, separate, or transit to adoption. Thus, the theoretical model will be used not only to have a

[6]The empirical literature suggests that decreasing the adoption penalty would increase adoption rates. Bishop and MacDonald (2024) analyze a policy change in the state of Minnesota that eliminated the financial penalty on adoption for children aged six and older, finding that the probability of adoption increased after the implementation. Argys and Duncan (2012) show that when the difference between the foster and adoption monthly payments decreases, a child's probability of adoption increases.

[7]The empirical specification and theoretical model can be used to study the effect of other observable characteristics of the child, such as gender, race, and ethnicity.

better understanding of the empirical facts aforementioned but it will also allow us to establish how the match transition of children is affected by match quality which is not observable to the econometrician.

To analyze how different forces interact in the agents' decisions of forming a foster match, separating a foster match, and transiting to permanency via adoption, I develop a dynamic matching model with search frictions (it takes time to find a match) and non-transferable utility (transfers are exogenously given). Children and parents can form two types of matches: foster (reversible) and adoption (irreversible). The setting assumes that (a) children are heterogeneous (with and without a disability), (b) agents must be foster matched before forming an adoption match, (c) parents receive a smaller per-period payoff when adoption matched than when foster matched, and (d) matches differ in their quality (low or high). Children and parents prefer matches of greater quality, and parents prefer children without a disability to children with a disability. The timing is as follows. Every period, when a child (unmatched or foster matched) and parent meet (unmatched only), agents draw a match quality. Before deciding whether to form a foster match, they observe only a noisy signal about this quality. A foster match forms if and only if both accept. If a new foster match forms, any old foster match dissolves. The uncertainty about the quality resolves once a foster match forms, and it remains constant throughout the match. After observing the match quality, agents decide whether to destroy the foster match (and become unmatched), transit to adoption, or remain foster matched.

Using the theoretical model, I disentangle the driving forces behind the aforementioned empirical facts, establish sufficient conditions on primitives for the empirical facts to emerge in equilibrium, and derive other equilibrium properties about match quality. Regarding separations, I find that the increase in the probability of foster match separation due to a disability (Fact 2) depends on two driving forces working in opposite directions. On the one hand, children with a disability are more likely (relative to children without a disability) to have a foster match separated after the uncertainty over the quality of the match is resolved, which itself makes them *more likely* to separate. On the other hand, children with a disability are less likely (relative to children without a disability) to form a new foster match, which itself makes them *less likely* to separate. Hence, Fact 2 suggests that the former driving force prevails in equilibrium: children with a disability are more likely to have a foster match separated because parents matched to these children are more likely to separate once the true quality is revealed. The dataset used for the analysis does not allow me to identify the reason for the separation so this gap is filled entirely by the theoretical model. In addition, as a model prediction, I find that high-quality foster matches are less likely to separate (relative to low-quality foster matches), and both of the driving forces behind having a foster match separated are aligned. That is, high-quality foster matches are less likely to separate after the uncertainty is resolved, and also less likely to separate due to the search for a better match.

Now, regarding adoption, I find that the decrease in the probability of being adopted due to a disability (Fact 1) arises for two reasons. First, I show that children with a disability (relative to children without a disability) are less likely to form a foster match because foster parents require greater signals to be willing to form a match with these children. Second, parents who are foster matched to children with a disability (relative to parents foster matched to children without a disability) have a greater incentive to remain in the reversible foster match and not transit to adoption. The reason is that the adoption penalty for children with a disability is greater (relative to children without a disability), and the probability that they separate the match in the future to form a new match is smaller (relative to children without a disability). Thus, parents adopting children with a disability accept a greater adoption penalty in exchange to eliminate a smaller probability of the match separating in the future. In this case, the intrinsic disadvantage (being less preferred by foster parents) faced by children with a disability exacerbates in the presence of the adoption penalty because it is not only more challenging for these children to find a parent willing to foster them, but due to this fact, parents fostering these children have less incentives to adopt because the threat of leaving the match in the future is smaller. Further, I find that parents in high-quality matches might have fewer incentives to adopt. The result is driven by the fact that children in foster matches of high-quality have fewer incentives to separate the foster match in the future. Hence, the adoption penalty not only exacerbates the intrinsic disadvantage faced by children with a disability, but also creates incentives for high-quality matches to not transit to adoption.

Related Literature. Most of the literature on dynamic matching with heterogeneous agents analyzes environments where matches do not reverse endogenously. Under this assumption, the literature has addressed issues regarding stability (Doval, 2022; Altinok, 2021), matching algorithms and its implications on welfare (Ünver, 2010; Anderson et al., 2015; Akbarpour et al., 2020; Baccara et al., 2020; Leshno, 2022), and positive assortative matching (Burdett and Coles, 1997; Eeckhout, 1999; Shimer and Smith, 2000; Chade, 2001, 2006; Smith, 2006). In these papers, agents face the trade-off of whether to form a match today or wait for a better partner. Now, if agents are allowed to form a match today and reverse it when a better partner arrives, an additional feature arises. In the presence of reversibility, agents must take into account that today's partner and the potential better partner of tomorrow might leave the match in the future. There is a small literature analyzing dynamic matching environments with reversibility of matches, but the focus is on stability and cooperative solution concepts (Damiano and Lam, 2005; Kurino, 2009; Kadam and Kotowski, 2018; Liu, 2021). This paper is more related to the literature on positive assortative matching by analyzing two-sided markets with search frictions, heterogeneous agents, and irreversible matches. My contribution adds to the literature on sorting along two dimensions. First, I allow for irreversible and reversible matches. Second, instead of addressing positive sorting, I estimate stylized facts present on the market and establish sufficient conditions for these patterns to arise in equilibrium.

In addition, I contribute to the narrow set of papers analyzing foster care as a matching market. Slaugh et al. (2015) studies the Pennsylvania Adoption Exchange program, a computational tool created to facilitate the adoption of children in foster care and make several recommendations to improve the success of adoptions. Olberg et al. (2021) constructs a dynamic search and matching model to compare two different search processes use by the child welfare agencies to identify potential adoption matches between parents and children. Lastly, Robinson-Cortés (2021) presents an empirical framework to study how children are assigned to foster homes using a confidential dataset, and uses the estimates to study different policy interventions. This paper departs from the previous literature mainly by considering both types of matches in one model, adoption (irreversible) and foster (reversible), allowing me to analyze a greater set of match transitions experienced by children.

Organization of the Paper. The rest of the paper is organized as follows. Section 2 presents the empirical analysis and the facts that motivate the theoretical model. Section 3 describes the theoretical environment, and introduces the equilibrium definition. Section 4 presents the equilibrium analysis and the conditions under which the equilibrium is consistent with the stylized facts, as well as the model predictions regarding match quality. Lastly, Section 5 concludes. All proofs are in the Appendix.

2 Empirical Analysis

I motivate the key features of the two-sided dynamic matching model described in the next section with an empirical analysis.[8] Using data describing the universe of children in the US foster care system over the period 2010 to 2016, I document new stylized facts about the match process between foster children and foster parents.

2.1 Data and Descriptive Statistics

I use the 6-month Foster Care Files from AFCARS,[9] an unbalanced panel of all children in the US foster care system between the federal fiscal years of 2010 and 2016. The data track a child upon entry into foster care until she exits, which could be due to reunification with birth-family, adoption, emancipation, guardianship, transfer to another agency, runaway, or death. If a child exits foster care, both the exit manner and date of exit are indicated. Additionally, the data include a rich set of variables describing the child,[10] such as gender, race and ethnicity, disability, whether the child is federally funded by Title IV-E,[11] date of birth, date of most recent entry into foster care, and date of termination of parental rights (if applicable).[12]

In the data, the disability variable, which is the focus of my empirical analysis, indicates whether a child has been clinically diagnosed with a disability, clinically diagnosed without a disability, or not yet diagnosed. For example, a disability includes conditions such as blindness, glaucoma, arthritis, multiple sclerosis, down syndrome, personality disorder, attention deficit, and anxiety disorder, among others. Unfortunately, data do not allow us to identify a specific disability, nor quantify either the number of disabilities or the severity. For the analysis, I say a **child has a disability** if she has been clinically diagnosed with at least one disability, and a child has no disability otherwise. In the majority of the cases, once a child enters the foster care system, a mandatory medical evaluation is performed;

[8]The empirical analysis does not seek to establish causality, but to obtain robust correlations controlling for a rich set of covariates.

[9]AFCARS is a federally mandated data collection system. All fifty US states and the District of Columbia are required to collect data on all children in foster care and all children adopted from foster care.

[10]Following Buckles (2013) and Brehm (2017), for all demographics I use the most recent record of each child since it updates all information.

[11]Title IV-E is a federal program through which states receive reimbursement of payments made on behalf of eligible children.

[12]To protect the confidentiality of the child, the date of birth is set to the 15th of the month and all dates are recoded to maintain consistent spans of time.

thus I assume that disabilities are pre-existing conditions.[13]

For each period (semester in the data) that a child remains in foster care, the data provide information about the last placement of the child during that period, including the start date of the placement. These placements are classified as: pre-adoptive home, non-relative foster home, relative foster home, group home, institution, supervised independent living, and runaway. Using these variables, I define a child as being **foster matched** in a given period if the child is placed in a pre-adoptive home, a non-relative foster home, or a relative foster home.[14] I define a child as being **unmatched** in a given period if the child is placed in a group home or institution.

To maintain a consistent estimation sample, I restrict the sample to children younger than age 16 whose parental rights have been terminated. The former restriction excludes older children who often exit through legal emancipation, and the latter is to ensure that children are eligible for adoption. I also restrict the sample to children who are either foster matched or unmatched. This leaves a full sample of $451,967$ children (sample A). Additionally, I create two subsamples. The first subsample (sample B) keeps only those child-period observations where the child is foster matched at the beginning of the period and still in foster care at the end of the period. The second subsample (sample C) keeps only those child-period observations where the child is unmatched at the beginning of the period and still in foster care at the end of the period. Table 1 presents summary statistics for the full sample and the two subsamples, and Table A1 presents these summary statistics conditioned on, the variable of interest, child's disability.

In Table 1 (sample A), children are, on average, almost 7 years old and have had their parental rights terminated for 17 months. Out of all children, 41 percent have been diagnosed with a disability. In a given period, 93 percent of children are foster matched, with the average duration of that match being 16 months. I say a child is **adopted** if she exits the system through adoption. On average, 28 percent of children are adopted in each period. I say a child **becomes unmatched** if conditional on being foster matched at the beginning of a period she is unmatched at the end of the same period. Conditional on starting the period foster matched (sample B), the probability that a child becomes unmatched is 2 percent. Now, I say a child **becomes foster matched** if conditional on being unmatched at the

[13] This is a strong assumption since disabilities could vary with the amount of time spent in a group home, or with the care provided by a foster parent. Ideally, we should consider disabilities as a potentially time-variant characteristic; however, data do not allow me to observe how a disability might evolve over time.

[14] It is important to mention that foster parents are not identifiable; when a child is placed in a foster home only family structure, foster parents' race and foster parents' year of birth are reported.

Table 1: Descriptive Statistics, All Samples

	Sample A $obs = 1,165,818$		Sample B $obs = 659,253$		Sample C $obs = 65,970$	
	Mean	sd	Mean	sd	Mean	sd
Adopted	0.28	0.45	-	-	-	-
Foster matched	0.93	0.25	1.00	0.00	0.00	0.00
Becomes foster matched	-	-	-	-	0.24	0.42
Becomes unmatched	-	-	0.02	0.14	-	-
Foster match separates	-	-	0.19	0.39	-	-
Age in years	6.80	4.43	6.81	4.36	12.17	2.80
Disabled	0.41	0.49	0.43	0.50	0.68	0.47
Male	0.53	0.50	0.52	0.50	0.63	0.48
White	0.43	0.50	0.42	0.49	0.44	0.50
Black	0.24	0.43	0.26	0.44	0.27	0.44
Hispanic	0.22	0.41	0.22	0.42	0.20	0.40
Title IV-E eligible	0.48	0.50	0.51	0.50	0.47	0.50
Months in foster care	34.87	24.38	34.86	24.86	53.10	36.72
Months since PRT*	17.09	22.62	16.23	22.15	41.99	36.86
ending in adoption	12.46	11.85	-	-	-	-
Months in current placement	16.06	15.72	17.31	16.32	10.85	13.99
foster matched	16.44	15.78	-	-	-	-

Notes: Data are from Adoption and Foster Care Analysis and Reporting System (AFCARS). Means and standard deviations are calculated for child-period observations. Sample A is the full sample containing all children younger than age 16 whose parental rights have been terminated and who are either foster matched or unmatched. Sample B and Sample C are subsamples of A. Sample B (sample C) keeps only those child-period observations such that the child is foster matched (unmatched) at the beginning of the period and still in foster care at the end of the period.
* PRT stands for Parental Rights Terminated.

beginning of a period she is foster matched at the end of the same period. The probability that a child becomes foster matched is 24 percent (sample C). It is important to highlight that the rates at which children experience match transitions are affected by the rates at which foster matches are separated. I say a child's **foster match separates** if conditional on being foster matched at the beginning of a period the child is no longer foster matched to the same parent at the end of the period.[15] Table 1 (sample B) shows that foster matches separate with probabil-

[15] Even though, foster parents are not identifiable, a variable recording the number of placements allows me to identify whether the child is being fostered by the same parent.

Table 2: Stylized Facts from Foster Care - Effect of Disability

	Adoption I	Foster match Separation II	Becomes Foster matched III	Becomes Unmatched IV
Disability γ	-0.059*** (0.005)	0.023*** (0.002)	-0.045*** (0.006)	0.011*** (0.001)
Mean of dependent variable	0.279	0.185	0.236	0.021
Number of child-period observations	1,165,818	659,253	65,970	659,253

Notes: Data are from Adoption and Foster Care Analysis and Reporting System (AFCARS). All specifications control for child's demographics, states indicators and period indicators. The first and second columns consider sample A, third and fifth columns use sample B, and the fourth column uses sample C. Standard errors are cluster at the state-period level and shown in parentheses. ***$P < 0.01$; **$P < 0.05$; *$P < 0.10$.

ity 19 percent. In practice, a separation can arise for different reasons such as the social worker decides to move the child to institutional care, the parent requests the removal of the child, or the social worker finds a more suitable foster parent for the child and decides to move the child. Unfortunately, the dataset does not contain this information.

2.2 Empirical Specifications and Stylized Facts

I estimate the impact of disability on four outcomes: (1) the probability that a child is adopted, (2) the probability that a foster match separates, (3) the probability that a child becomes foster matched, and (4) the probability that a child becomes unmatched. For each outcome, I estimate the following linear probability model:

$$y_{ijt} = \alpha + \gamma\ disability_i + \beta X_i + \theta Z_{it} + \xi_j + \lambda_t + \epsilon_{ijt} \tag{1}$$

where y_{ijt} is an indicator for the outcome of child i in state j at period t. $disability_i$ is an indicator equal to one if child i has a disability and zero otherwise. X_i is a vector of time-invariant characteristics of child i such as gender, race, ethnicity, and whether the child is federally funded through Title IV-E. Z_{it} is a vector of time-varying characteristics of child i including age in months, number of months in foster care, and number of months since parental rights have been terminated. I include a vector of period fixed-effects λ_t to control for time-trends, and a vector of state fixed-effects ξ_j to control for unobserved state characteristics.

2.2.1 Fact 1: Disability Decreases the Probability of Being Adopted

The adoption rates of children with and without a disability are 0.22 and 0.32, respectively (see Table A1). To evaluate the significance of this effect conditional on other demographics, I use sample A to estimate Equation 1 where the dependent variable y_{ijt} is equal to one if child i in state j is adopted in period t and zero if she either remains in foster care or exits through any other manner. Table 2 column I shows that children with a disability are 6 percent less likely to be adopted than children without a disability.

As many states require parents to foster a child before an adoption can take place, the fact that children with a disability are less likely to be adopted might be driven by the fact that these children are less likely to be fostered in the first place. To analyze this, I estimate a version of Equation 1where the dependent variable y_{ijt} is redefined to take the value of one if child i in state j is foster matched in period t and zero otherwise. As in adoption, the coefficient on disability is negative (see Table A2). While this is suggestive, the theoretical model will allow me to show that children with a disability are less likely to be adopted not only because they are less likely to be foster matched, but they are also less likely to transit from a foster match to adoption.

2.2.2 Fact 2: Disability Increases the Probability of Foster Match Separation

From the data, foster matches constituted by children with and without a disability separate at rates 0.19 and 0.18, respectively (see Table A1). Using sample B, I estimate Equation 1 where the dependent variable y_{ijt} is equal to one if child i in state j has her foster match separated in period t and zero otherwise. Here, the vector Z_{it} includes the number of months that the child has been in her current foster match and what type of foster match it is (i.e., whether a pre-adoptive home, non-relative foster home or relative foster home).

Table 2 column II shows that children with a disability are 2 percent more likely to have their foster match separated than children without a disability. Even though, the dataset does not allow to identify the reason of the separation, the theoretical model will separately identify two types of separations: (1) the child transits from foster matched to unmatched i.e. from foster home to institutional care, and (2) the child transits from a foster match to another foster match i.e. from foster home to foster home. Furthermore, I will show that these two forces work on opposition directions: children with a disability are more likely to experience the first type of separation, and less likely to experience the second type.

2.2.3 Fact 3: Disability Decreases the Probability of Becoming Foster Matched

The rates of foster match formation (conditional on starting the period unmatched) of children with and without a disability are 0.22 and 0.28, respectively (see Table A1). To study the effect of disability on the probability of becoming foster matched, I use sample C to estimate Equation 1 where the dependent variable y_{ijt} equal one indicates that child i in state j becomes foster matched in period t and zero otherwise. In this specification, the vector Z_{it} additionally includes the number of months that the child has been in her current unmatched state and where she is currently living (i.e., whether a group home or institution).

Table 2 column III shows that disability decreases the probability of becoming foster matched by 5 percent. That is, children with a disability are less likely to become foster matched than children without a disability. The theoretical model will show that this probability is driven by the fact that disability decreases the probability that a child finds a parent willing to foster her, and if they do, disability increases the probability that the foster match is later on destroyed.

2.2.4 Fact 4: Disability Increases the Probability of Becoming Unmatched

From the data, the rates of unmatched formation (conditional on starting the period foster matched) of children with and without a disability are 0.03 and 0.01, respectively (see Table A1). Here, I use sample B to estimate Equation 1 where the dependent variable y_{ijt} equal one indicates child i in state j becomes unmatched in period t and zero otherwise. As in the previous estimation, Z_{it} includes the number of months that the child has been in her current foster match and the type of foster match.

As we can see from Table 2 column IV, disability increases the probability of becoming unmatched. In the model, the probability of becoming unmatched will depend on the rate at which foster matches separate and the probability that a child finds a parent willing to foster her. Thus, behind this stylized fact, there are driving forces working on opposite directions, as in the case of separations.

3 Model

In this section, I develop a search and matching model to analyze how different incentives interact in the agents' decisions over match formation and separation. With the data available is not possible to make any statement regarding what type of matches, in terms of match quality, are more likely to form, separate, or transit to adoption. Thus, the theoretical model will be used not only to have a better understanding of the empirical facts estimated in Section 2 but it will also allow us to establish how the match transition of children is affected by match-quality which is not observable to the econometrician.

3.1 Environment

Time is discrete with an infinite-horizon. One side of the market is populated by **children** who differ in an observable attribute $x \in X = \{x_1, x_2\}$ where x_1 denotes a child with a disability, x_2 indicates a child without a disability, and $x_1 < x_2$. Each period, a strictly positive mass of children ρ enters the market and each child draws an attribute from a full support probability distribution $l(x)$. The other side of the market is constituted by homogeneous **parents**. The mass of parents out of the market is strictly positive, and parents make entry and exit decisions each period.

Children and parents who are in the market can be unmatched or matched. Let $u_t^p \geq 0$ denote the endogenous distribution of unmatched parents in the market, and $u_t^c(x)$ denote the endogenous distribution of unmatched children in the market. Matches are one-to-one, formed between children and parents, and heterogeneous in **quality** denoted as $q \in Q = \{q_1, q_2\}$ where $q_1 < q_2$.[16] Further, I define two types of matches: **foster matches** (reversible) and **adoption matches** (irreversible). Agents who form a foster match (hereafter *f-match*) remain in the market, while agents who form an adoption match (hereafter *a-match*) leave the market. Let $m(x, q)$ denote the endogenous distribution over f-matches. Thus, the aggregate state of the market is summarized by $\phi = (u^p, u^c, m)$.

All agents are risk-neutral and discount future at rate $\beta \in (0, 1)$. Payoffs for unmatched children are normalized to zero. For children who are f-matched or a-matched, payoffs are given by the real-valued function $b^c(x, q, z)$ where $z \in \{f, a\}$, $z = f$ indicates an f-match, $z = a$ indicates an a-match, and $a > f$. I assume that children' payoff function satisfies the following:

Assumption 1 (Payoffs of Children). *(a) $b^c(x, q, a) > b^c(x, q, f) \geq 0$ for all (x, q);*

(b) $b^c(x, q, z)$ is decreasing in x;

(c) $b^c(x, q, z)$ is increasing in q;

[16]Match quality captures other factors affecting the match independent of the child's attribute, such as the emotional bond between the child and parent, or the relationship between the parent and the child's birth family.

(*d*) $b^c(x, q_2, f) > b^c(x, q_1, a)$;

(*e*) $b^c(x, q, z)$ *is supermodular in* (x, z);

(*f*) $b^c(x, q, z)$ *is submodular in* (x, q); *and*

(*g*) $b^c(x_1, q_2, f) - b^c(x_1, q_1, a) > b^c(x_2, q_2, f) - b^c(x_2, q_1, a)$.

Assumption 1(a) captures that children are better-off with a foster parent than in institutional care, and better-off when adopted than fostered. 1(b) reflects that children with a disability benefit more from the family environment and emotional stability provided by foster and adoption. The intuition that children are better-off in high-quality matches is addressed in 1(c). Assumption 1(d) states that children prefer to be f-matched when the quality is high than a-matched when the quality is low. 1(e) imposes that the gain of being adopted is greater for children without a disability, and 1(f) captures that the gain of being in high-quality matches is greater for children with a disability. Lastly, assumption 1(g) implies that the gain of being in an f-match of high-quality versus being in an a-match of low-quality is greater for children with a disability.

Payoffs for parents out of the market are normalized to zero. Parents incur on a per-period cost $k > 0$ to hold a license and stay in the market. Parents who are f-matched or a-matched receive payoffs according to the real-valued function $b^p(x, q, z)$. I assume that parents' payoff function satisfies the following:

Assumption 2 (Payoffs of Parents). (*a*) $b^p(x, q, f) > b^p(x, q, a)$ *for all* (x, q);

(*b*) $b^p(x, q, z)$ *is increasing in* x;

(*c*) $b^p(x, q, z)$ *is increasing in* q;

(*d*) $b^p(x, q, z)$ *is log-supermodular in* (x, z); *and*

(*e*) $b^p(x, q, z)$ *is log-submodular in* (q, z).

Assumption 2(a) reflects the presence of the adoption penalty. 2(b) captures the intuition that parents prefer children without a disability to children with a disability. 2(c) reflects that parents in high-quality matches benefit more from fostering/adopting than parents in low-quality matches. Now, the term $1 - \frac{b^p(x,q,a)}{b^p(x,q,f)}$ represents the **adoption penalty.** Assumption 2(d) states that the adoption penalty is greater for children with a disability. Lastly, assumption 2(e) imposes that the adoption penalty is increasing in the match quality.

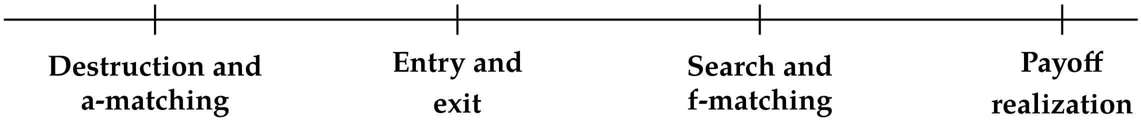

Figure 1: Timeline

Figure 1 exhibits the timeline within a period. Each period is divided into four stages:

1. Search and f-matching stage. Children search when unmatched or f-matched, and parents search only when unmatched. Meetings are stochastic and can be described in terms of the market tightness $\theta \in \mathbb{R}_+$ (i.e. parents-to-children ratio):

$$\theta = \frac{u^p}{\sum_x u^c(x) + \sum_q m(x,q)}.$$

A child meets a parent with probability $\pi^c(\theta)$ which is a strictly increasing and strictly concave function such that $\pi^c(0) = 0$. Similarly, a parent meets a child with probability $\pi^p(\theta)$ which is a strictly decreasing and convex function such that $\pi^p(\theta) = \frac{\pi^c(\theta)}{\theta}$ and $\pi^p(0) = 1$. Next, when a child and parent meet, a match quality q is drawn from the full support probability distribution $h(q)$. A match quality is constant through the duration of the f-match, and learned through experience. Before forming an f-match, agents observe a noisy signal $s \in S = \{s_1, s_2\}$ generating a full support conditional probability distribution $g(q|s)$ such that if $s' > s$ then $G(q|s') \leq G(q|s)$. After observing the noisy signal, agents announce simultaneously 'foster' or 'reject'. An f-match is formed if and only if both agents announce foster. If a new f-match is formed, any old f-match dissolves.

2. Payoff realization stage. Agents in newly formed f-matches perfectly observe the quality q. Once a match quality is complete information, payoffs received during the remaining duration of the f-match are known.

3. Destruction and a-matching stage. A child x is adopted by a relative with exogenous probability $\delta_x \in (0,1)$ where $\delta_{x_2} \geq \delta_{x_1}$.[17] The f-match separates, if a child is adopted exogenously. Now, if the f-match remains, then child and parent announce simultaneously 'adoption', 'destroy', or 'remain'. An f-match destroys if at least one agent announces destroy, and an a-match takes place if and only if both agents announce adoption. If an f-match destroys, the parent remains unmatched that period and the child searches. Agents who form an a-match receive adoption

[17] In some cases, relatives reach out when they learn about the situation and request to adopt the child. Child welfare agencies have strong preferences for relatives.

payoffs to perpetuity, and I assume that q remains constant when transitioning from f-matched to a-matched. Children adopted by a relative receive $b^c(x, q_2, a)$ to perpetuity.

4. Entry and exit stage. A mass of new children enters the market and parents make entry/exit decisions. Parents and children who enter the market remain unmatched that period. Agents who formed an a-match during the previous stage leave the market, and only unmatched parents can decide to exit the market.

I restrict attention to stationary pure symmetric Markov strategies. Strategies depend on the aggregate state of the market ϕ, and to simplify notation I suppress it. I refer to a parent f-matched to child x with match quality q as **parent (x,q)**. For each parent, a strategy consists of the tuple (in, out, f^p, d^p, a^p) where $in \in \{no, yes\}$ is the entry strategy, $out \in \{no, yes\}$ is the exit strategy, $f^p(x, s) : X \times S \to \{reject, foster\}$ is the decision to form an f-match with child x after observing signal s, $d^p : X \times Q \to \{0, 1\}$ is the decision to destroy the f-match such that $d^p(x, q) = 1$ when parent (x, q) announces destroy, and $a^p : X \times Q \to \{0, 1\}$ is the decision to form an a-match such that $a^p(x, q) = 1$ when parent (x, q) announces adoption. Now, refer to child x f-matched with quality q as **child (x,q)**, and refer to an unmatched child x as **child ($\mathbf{x,q_0}$)**. To make reference to a child's match status, I define an auxiliary set $\bar{Q} = Q \cup \{q_0\}$. For each child x, a strategy consists of the triple (f^c, d^c, a^c) where $f^c : X \times \bar{Q} \times S \to \{reject, foster\}$ is the decision to form a new f-match after child $(x, \bar{q})$ observes signal s, and $d^c : X \times Q \to \{0, 1\}$ and $a^c : X \times Q \to \{0, 1\}$ are the destruction and adoption decisions, respectively.

Lastly, let $d(x, q) = d^c(x, q) + (1 - d^c(x, q))d^p(x, q)$ and $a(x, q) = a^c(x, q)a^p(x, q)$ denote the joint destruction and adoption decisions of an f-match (x, q), and define the f-matching correspondence as follows:

Definition 1. *A **foster-matching correspondence** is a map $\mathcal{M} : X \times \bar{Q} \mapsto S$ such that $s \in \mathcal{M}(x, \bar{q})$ if and only if **(i)** child $(x, \bar{q})$ is willing to form an f-match after observing signal s, and **(ii)** unmatched parent is willing to form an f-match after meeting child x and observing signal s.*

3.2 Value Functions

3.2.1 Value Functions for Children

Let $\mathcal{C}(x, \bar{q})$ denote the value function for child $(x, \bar{q})$ at the end of a period, and define $\hat{\mathcal{C}}(x, \bar{q})$ as the search value for child $(x, \bar{q})$ at the beginning of the search and f-matching stage. The search value function is specified by Equation 2. At the beginning of the search and f-matching stage, child $(x, \bar{q})$ meets a parent with endogenous probability $\pi^c(\theta)$. If no meeting takes place, status-quo is preserved and she receives the continuation value $\mathcal{C}(x, \bar{q})$. If a meeting takes place, a noisy signal s is realized where $f(s)$ is the probability distribution over signals derived from $h(q)$ and $g(q|s)$. If at least one agent announces reject after observing s, then the status-quo is preserved. If both agents announce foster after observing s, then the child receives the conditional expected value $\mathbb{E}_s[\mathcal{C}(x, q)] = \sum_q \mathcal{C}(x, q) g(q|s)$.

$$\hat{\mathcal{C}}(x, \bar{q}) = \Big(1 - \pi^c(\theta) \sum_{\mathcal{M}(x,\bar{q})} f(s)\Big)\mathcal{C}(x, \bar{q}) + \pi^c(\theta) \sum_{\mathcal{M}(x,\bar{q})} \mathbb{E}_s[\mathcal{C}(x, q)] f(s) \qquad (2)$$

Thus, child $(x, \bar{q})$ announces foster after observing s if and only if the conditional expected value of forming a new f-match is greater than the continuation value of the status-quo i.e. $\mathbb{E}_s[\mathcal{C}(x, q)] \geq \mathcal{C}(x, \bar{q})$. For child x who is unmatched at the end of a period, the value function is:

$$\mathcal{C}(x, q_0) = \beta\delta_x \frac{b^c(x, q_2, a)}{1 - \beta} + \beta(1 - \delta_x)\hat{\mathcal{C}}(x, q_0) \qquad (3)$$

Now, consider a child x f-matched with quality q at the end of a period. Child (x, q)'s value function is specified by Equation 4. In the current period, she receives the f-match payoff $b^c(x, q, f)$. At the beginning of the next period, she is adopted by a relative with probability δ_x. If the f-match remains, child and parent decide between transit to an a-match, destroy the f-match, or remain f-matched. In each case, child (x, q)'s possible continuation values are $\frac{b^c(x,q,a)}{1-\beta}$, $\hat{\mathcal{C}}(x, q_0)$, and $\hat{\mathcal{C}}(x, q)$ respectively.

$$\begin{aligned}\mathcal{C}(x, q) = b^c(x, q, f) + \beta\delta_x \frac{b^c(x, q_2, a)}{1 - \beta} + \beta(1 - \delta_x)\Bigg[d^p(x, q)\hat{\mathcal{C}}(x, q_0) \\ + a^p(x, q) \max\left\{\frac{b^c(x, q, a)}{1 - \beta}, \hat{\mathcal{C}}(x, q_0), \hat{\mathcal{C}}(x, q)\right\} \\ + \Big(1 - d^p(x, q) - a^p(x, q)\Big) \max\Big\{\hat{\mathcal{C}}(x, q_0), \hat{\mathcal{C}}(x, q)\Big\}\Bigg] \qquad (4)\end{aligned}$$

Thus, child (x, q) chooses adoption if and only if the value of being adopted is greater than the value of continue searching while unmatched and the value of continue searching while f-matched when the quality is q. Hence, a child faces the following trade-off: receive a greater adoption payoff but forgo the opportunity of finding a 'better' match. Similarly, child (x, q) chooses destroy if and only if the value of searching while unmatched is greater than the value of being adopted and the value of continue searching while f-matched. Hence, when a child decides to destroy a f-match, she is destroying a 'bad' match.

3.2.2 Value Functions for Parents

Let $\mathcal{P}^u$ denote the value function for an unmatched parent and $\mathcal{P}(x,q)$ denote the value function for parent (x,q). For an unmatched parent, the value function is presented in Equation 5. In the current period, the unmatched parent incurs in the per-period cost k of holding a license. Next, the parent decides between stay or exit the market. If she exits her payoff is zero, and if she stays she meets a child with probability $\pi^p(\theta)$. When no meeting takes place, the parent remains unmatched. When a meeting takes place, a child is drawn from the endogenous probability distribution $\hat{m}(x,\bar{q})$ derived from u^c and m (for detail see Appendix B.1). After meeting child $(x,\bar{q})$, agents observe some signal. If at least one agent announces reject, then the parent remains unmatched. If both announce foster, then the parent receives $\mathbb{E}_s[\mathcal{P}(x,q)]$.

$$\mathcal{P}^u = \max\left\{0, \frac{-k + \beta\pi^p(\theta)\sum_{\mathcal{M}(x,\bar{q})}\sum_{x,\bar{q}}\mathbb{E}_s[\mathcal{P}(x,q)]\hat{m}(x,\bar{q})f(s)}{1-\beta\Big(1-\pi^p(\theta)\sum_{\mathcal{M}(x,\bar{q})}\sum_{x,\bar{q}}\hat{m}(x,\bar{q})f(s)\Big)}\right\} \tag{5}$$

Thus, an unmatched parent forms an f-match with child $(x,\bar{q})$ after observing signal s if and only if $\mathbb{E}_s[\mathcal{P}(x,q)] \geq \mathcal{P}^u$.

For parent (x,q), the value function is Equation 6. In this period, she receives $b^p(x,q,f)$. Next period, she becomes unmatched with probability δ_x. If the f-match remains, child and parent decide between transit to adoption, destroy the f-matched or remain f-matched.

$$\begin{aligned}
\mathcal{P}(x,q) &= b^p(x,q,f) + \beta\delta_x\mathcal{P}^u + \beta(1-\delta_x)\Bigg[d^c(x,q)\mathcal{P}^u \\
&+ a^c(x,q)\max\left\{\frac{b^p(x,q,a)}{1-\beta}, \mathcal{P}^u, \Big(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\Big)\mathcal{P}(x,q) + \pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\mathcal{P}^u\right\} \\
&+\Big(1-d^c(x,q)-a^c(x,q)\Big)\max\left\{\mathcal{P}^u, \Big(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\Big)\mathcal{P}(x,q)+\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\mathcal{P}^u\right\}\Bigg]
\end{aligned} \tag{6}$$

Hence, when parent (x,q) is deciding to adopt, she faces the following trade-off: forgo part of the per-period payoff in exchange to eliminate the likelihood that the f-match is destroyed in the future.

3.3 Aggregate State of the Market

The distribution of unmatched parents in the market depends on the entry and exit strategies of parents. Thus, the stationary mass of unmatched parents u^p satisfies the following inequality:

$$\pi^p\left(\frac{u^p}{\sum_x u^c(x) + \sum_q m(x,q)}\right) \leq \frac{k}{\beta \sum_{\mathcal{M}(x,\bar{q})} \sum \mathbb{E}_s[\mathcal{P}(x,q)]\hat{m}(x,\bar{q})f(s)} \tag{7}$$

with equality if u^p is strictly positive. For distributions $u^c(x)$ and $m(x,q)$ to be time invariant, the mass destruction and mass creation must exactly balance (for detail see Appendix B.2).

3.4 Definition of Equilibrium

See Appendix C.

4 Theoretical Analysis

I first derive equilibrium properties and identify the driving forces behind the empirical results estimated in Section 2. Afterwards, I use these properties to ensure that the empirical facts arise in equilibrium and carry out model predictions regarding match quality.

4.1 Equilibrium Analysis

The analysis focuses on foster care equilibria with a positive mass of parents in the market i.e $u^p > 0$ which implies that $\mathcal{P}^u = 0$ (from Equations 5 and 7). Moreover, I assume that for each child, there is at least one signal such that parents receive a positive expected foster payoff.

Assumption 3. *For each x, there exists $\hat{s}$ such that $\mathbb{E}_{\hat{s}}[b^p(x,q,f)] \geq 0$.*

Proposition 1 presents how the destruction strategy of f-matches varies with disability status and match quality. Item (i) establishes that, after the uncertainty over the quality of the match is resolved, f-matches involving children with a disability destroy more than f-matches involving children without a disability. Formally, fixing q, if the f-match (x_2, q) is destroyed then the f-match (x_1, q) is also destroyed. Recall that, an f-match can be destroyed by either the child or the parent, $d(x,q) = d^c(x,q) + (1 - d^c(x,q))\, d^p(x,q)$. By Assumption 1(a), it follows that children never destroy an f-match. Thus, in equilibrium, the destruction is driven by parents, which is consistent with the anecdotal evidence suggesting that when a child moves from foster home to institutional care is generally due to the request of the foster parent. Now, by Assumption 2(b), it follows that if $d^p(x_2, q) = 1$ then $d^p(x_1, q) = 1$ for all q. In item (ii), I show that, if the f-match (x, q_1) is destroyed then f-match (x, q_2) is also destroyed. In words, if a parent f-matched to child x when the quality is q_2 is not willing to continue providing care, then a parent f-matched to child x when the quality is q_1 is also not willing to continue providing care. This follows from Assumption 2(c).

Proposition 1 (**Destruction**). *Assume children' payoffs satisfy Assumption 1(a), and parents' payoffs satisfy Assumptions 2(a)-(c). Then, in any foster care equilibrium:*

(*i*) *f-match destruction is greater for children* ***with a disability****,*

$$d(x_1, q) \geq d(x_2, q), \text{ for all } q.$$

(*ii*) *f-match destruction is greater for* ***low-quality*** *matches,*

$$d(x, q_1) \geq d(x, q_2), \text{ for all } x.$$

Proof. See Appendix D.1. □

To establish the empirical facts, I will ensure that a parent's strategies satisfy the following:

(1) if a parent is willing to form an f-match with child x_1 after observing signal s, then she is also willing to form an f-match with child x_2 after observing s.

(2) if a parent is willing to adopt child x_1 when the quality is q, then she is also willing to adopt child x_2 when the quality is q.

Since (1) might contradict (2), I impose Assumption 4 which allows me to characterize a parent's f-match formation strategies using the per-period payoffs. This assumption ensures that, if the conditional expected *payoff* received by a parent f-matched to child (x, q) is negative, then the conditional expected *value* of being f-matched to child (x, q) is also negative.

Assumption 4. *For each (s, x), if $\mathbb{E}_s[b^p(x, q, f)] < 0$ then the following condition on primitives holds:*

$$\mathbb{E}_s\left[b^p(x,q,f) + \beta\,(1-\delta_x)\sum_q \max\left\{\frac{b^p(x,q,a)}{1-\beta}\,,0\,,\frac{b^p(x,q,f)}{1-\beta\,(1-\delta_x)}\right\} g(q|s)\right] < 0$$

Proposition 2 exhibits how the formation of f-matches involving unmatched children varies with disability status and match quality. Recall that f-matches must be mutually agreed upon, that is, $s \in \mathcal{M}(x, q_0)$ if and only if $s \in F^p(x)$ and $s \in F^c(x, q_0)$. By Assumption 1(a), it follows that children always announce foster after observing signal s. Intuitively, as the law requires, children are placed in foster family homes whenever possible. Thus, the formation of an f-match depends on the parent's decision. In item (i), I show that conditional on observing signal s, if a parent is willing to foster a child with a disability, then she must also be willing to foster a child without a disability i.e. if $s \in F^p(x_1)$ then $s \in F^p(x_2)$. This follows from Assumption 2(b). In words, children with a disability are less likely to find a parent willing to foster them. In item (i), I state that if a parent announces foster after meeting child x and observing signal s_1, then she also announces foster after observing signal s_2. The result follows from Assumption 2(c). Since $G(q|s_1)$ first-order stochastically dominates $G(q|s_2)$, it follows that the conditional expected value received by a parent when fostering a child is increasing in the signal.

Proposition 2 (**F-match formation involving unmatched children**). *Assume children' payoffs satisfy Assumption 1(a), and parents' payoffs satisfy Assumptions 2(b)-(c), 3 and 4. Then, in any foster care equilibrium:*

(*i*) *f-match formation is smaller for unmatched children **with a disability**,*

$$\mathcal{M}(x_1, q_0) \subseteq \mathcal{M}(x_2, q_0).$$

Moreover, $\mathcal{M}(x, q_0)$ is non-empty for all x.

(*ii*) *f-match formation is greater for **high-signals**,*

$$s_1 \in \mathcal{M}(x, q_0) \text{ implies } s_2 \in \mathcal{M}(x, q_0), \text{ for all } x.$$

Proof. See Appendix D.2. □

Proposition 3 exhibits how f-match formation involving f-matched children varies with disability status and match quality. Item (i) states that children without a disability are more likely to form a new f-match than children with a disability. The result is driven by the parents' decision: children without a disability are more demanded by foster parents. Item (ii) shows that low-quality matches are more likey to form new f-matches than high-quality matches. The result is driven by the children' decision. By Assumption 1(d), children value more quality than the adoption status, thus they have no incentives to separate high-quality matches.

Proposition 3 (**F-match formation involving f-matched children**). *Assume children' payoffs satisfy Assumptions 1(a),(c)-(d), and parents' payoffs satisfy Assumptions 2(a)-(c), 3 and 4. Then, in any foster care equilibrium:*

(*i*) *f-match formation is smaller for children **with a disability**,*

$$\sum_{\mathcal{M}(x_2,q)} f(s) \geq \sum_{\mathcal{M}(x_1,q)} f(s), \text{ for all } q.$$

(*ii*) *f-match formation is greater when the old match is **low-quality**,*

$$\sum_{\mathcal{M}(x,q_1)} f(s) \geq \sum_{\mathcal{M}(x,q_2)} f(s) = 0, \text{ for all } x.$$

Proof. See Appendix D.3. □

Due to Proposition 3(i), children with a disability are more willing to announce adoption after observing a low-quality match because their search opportunities are smaller. However, the intuition suggests that social workers might be pickier when searching for an adoptive parent for a child with a disability since these children benefit more from higher quality matches. Thus, to ensure that this intuition arises in equilibrium, I impose stronger conditions in Assumption 5. These conditions will help to ensure that if a child with a disability is willing to give up the opportunity of continue searching for a high-quality match, then children without a disability will also be willing to give up this opportunity.

Assumption 5. *Children' payoffs satisfy the following:*

(*a*) $\frac{\delta_{x_1}}{\delta_{x_2}} > \frac{b^c(x_2,q_2,a)-b^c(x_2,q_1,a)}{b^c(x_1,q_2,a)-b^c(x_1,q_1,a)}$

(*b*) $\frac{\{b^c(x_1,q_2,f)-b^c(x_1,q_1,a)\}(1-\beta(1-\delta_{x_1}))-\{b^c(x_1,q_2,a)-b^c(x_1,q_2,f)\}\beta(1-\delta_{x_1})}{1-\beta(1-\delta_{x_1})} > b^c(x_2,q_2,f) - b^c(x_2,q_1,a)$

(*c*) $\frac{\{b^c(x_1,q_2,f)-b^c(x_1,q_1,a)\}\beta\delta_{x_1}-\{b^c(x_1,q_2,a)-b^c(x_1,q_2,f)\}(1-\beta)}{g(q_2|s_1)} > b^c(x_2,q_2,f)(1-\beta) + b^c(x_2,q_2,a)\beta - b^c(x_2,q_1,a)$

Proposition 4 exhibits how adoption outcomes vary with disability status and match quality. Item (i) states that children with a disability transit to adoption less than children without a disability. Both parents' and children's decisions drive the result. Item (ii) shows that if the probability that the child leaves the f-match is sufficiently low, then high-quality matches do not transit to adoption due to the parents' decision. Thus, high-quality matches transit to adoption less than low-quality matches.

Proposition 4 (**Adoption**)**.** *Assume children' payoffs satisfy Assumptions 1(a)-(g) and 5(a)-(c), and parents' payoffs satisfy Assumptions 2(a)-(e), 3 and 4. Then, in any foster care equilibrium:*

(*i*) *a-match formation is smaller for children* ***with a disability***,

$$a(x_2, q) \geq a(x_1, q), \textit{for all } q.$$

(*ii*) *a-match formation is greater for* ***low-quality*** *matches,*

$$a(x, q_1) \geq a(x, q_2), \textit{for all } x.$$

Proof. See Appendix D.4. □

4.2 Empirical Facts and Model Predictions

Now, I establish sufficient conditions on primitives such that the empirical results estimated in Section 2 emerge in equilibrium, and analyze the role of match quality in the empirical facts. From now on, I assume all the assumptions specified previously hold.

4.2.1 Probability of Being Adopted

Consider child $(x, \bar{q})$ at the beginning of a period, and let $A(x, \bar{q})$ denote the probability that she becomes a-matched next period specified as:

$$A(x, q_0) = \delta_x + (1 - \delta_x)\, \pi^c(\theta) \sum_{\mathcal{M}(x,q_0)} f(s) \sum_{q'} g(q'|s) \Big[\delta_x + \big(1 - \delta_x\big) a(x, q')\Big]$$

and

$$\begin{aligned} A(x, q) &= \delta_x \\ &+ (1 - \delta_x) \Bigg\{ a(x, q) + d(x, q)\, \pi^c(\theta) \sum_{\mathcal{M}(x,q_0)} f(s) \sum_{q'} g(q'|s) \Big[\delta_x + \big(1 - \delta_x\big) a(x, q')\Big] \\ &\quad + \Big(1 - a(x, q) - d(x, q)\Big)\, \pi^c(\theta) \sum_{\mathcal{M}(x,q)} f(s) \sum_{q'} g(q'|s) \Big[\delta_x + \big(1 - \delta_x\big) a(x, q')\Big] \Bigg\} \end{aligned}$$

In the first case, the probability that a child (x, q_0) is adopted endogenously depends on the child forming an f-match during the search and f-matching stage, and both agents announcing adoption after observing some quality q. In the second case, the probability that child (x, q) is adopted endogenously can be decomposed in three events: (a) f-match (x, q) transits to adoption, (b) f-match (x, q) destroys and the unmatched child transits to an a-match with another parent, and (c) the f-match (x, q) remains but the child finds a new f-match and transits to an a-match with another parent.

Corollary 1. *In any foster care equilibrium, the probability of being adopted is:*

(i) *smaller for children* ***with a disability*** *if* $\frac{\delta_{x_2}-\delta_{x_1}}{1-\delta_{x_1}} > \overline{\pi}$ *holds.*

(ii) *greater for* ***low-quality*** *matches if* $b^p(x, q_1, a) > 0$ *and* $\frac{b^p(x,q_2,a)}{b^p(x,q_2,f)} \leq \frac{1-\beta}{1-\beta(1-\delta_x)}$ *hold.*

Proof. See Appendix E.1. □

Corollary 1(i) exhibits sufficient conditions for Fact 1 to arise in equilibrium. I say that children with a disability are less likely to be adopted if $A(x_2, \bar{q}) \geq A(x_1, \bar{q})$ holds for all $\bar{q}$. Loosely speaking, children with a disability are less likely to form an f-matched, and if they do, they are less likely to transit to adoption.

Corollary 1(ii) presents the impact of match quality on the probability of being adopted. I say that the probability of being adopted is decreasing in match quality if $A(x, q_1) \geq A(x, q_2)$ holds for all x. In the presence of the adoption penalty, when the exhibited conditions are satisfied, high-quality matches are less likely to transit to an a-match than low-quality matches. Intuitively, if the separation of high-quality matches is low enough, then parents have no incentives to choose adoption.

4.2.2 Probability of Foster Match Separation

Consider child (x, q) at the beginning of a period, and let $D(x, q)$ denote the probability that the f-match separates within a period:

$$D(x, q) = (1-\delta_x)(1-a(x, q))\left[d(x, q) + \big(1-d(x, q)\big)\ \pi^c(\theta) \sum_{\mathcal{M}(x,q)} f(s)\right]$$

The probability that an f-match (x, q) separates is decomposed in two events. First, f-match (x, q) destroys during the destruction and a-matching stage. Second, f-match (x, q) remains but, during the search and f-matching stage, child x forms a new f-match with some parent after observing signal s.

Corollary 2. *In any foster care equilibrium, the probability of foster match separation is:*

(*i*) *greater for children* ***with a disability*** *if* $\frac{\delta_{x_2} - \delta_{x_1}}{1 - \delta_{x_1}} \geq f(s_1)$ *holds.*

(*ii*) *greater for* ***low-quality*** *matches if* $a(x, q_1) = 0$ *and* $a(x, q_2) = 0$ *hold.*

Proof. See Appendix E.2. □

Corollary 2(i) exhibits sufficient conditions for Fact 2 to arise in equilibrium. I say that children with a disability are more likely to have a foster match separation if $D(x_2, q) \geq D(x_1, q)$ holds for all q. This depends on two forces working on opposite directions, and the empirical result sheds light on which of the driving forces prevails in equilibrium. On the one hand, Proposition 1(i) shows that children with a disability are more likely to have an f-matched destroyed, which by itself makes them more likely to separate On the other hand, Proposition 3(i) shows that children with a disability are less likely to form a new f-match, which by itself makes them less likely to separate. Hence, foster separation involving children with a disability are mainly driven by the uncertainty on the quality of the match, while foster separations affecting children without a disability are driven mostly by the search to improve the match quality.

Corollary 2(ii) presents sufficient conditions such that the probability of foster match separation is decreasing in match quality, $D(x, q_1) \geq D(x, q_2)$ for all x. In this case, the driving forces behind separation are aligned. Specifically, as long as agents' payoffs are increasing in quality (along with other conditions), the probability of separation is decreasing in match quality.

4.2.3 Probability of Becoming Foster Matched

Consider child (x, q_0) at the beginning of a period, then the probability that child x becomes f-matched next period is denoted as $M(x)$:

$$M(x) = (1 - \delta_x) \left[\pi^c(\theta) \sum_{\mathcal{M}(x, q_0)} f(s) \sum_q g(q|s)\,(1 - \delta_x)\,\big(1 - d(x, q)\big) \right]$$

Corollary 3 describes the sufficient conditions for Fact 3 to arise in equilibrium. I say that children with a disability are less likely to become foster matched if $M(x_2) \geq M(x_1)$ holds. In this case, children with a disability are less likely to form an f-match, and if they form an f-match, children with a disability are more likely to have it destroyed.

Corollary 3. *In any foster care equilibrium, the probability of becoming foster matched is smaller for children* ***with a disability****.*

Proof. See Appendix E.3. □

4.2.4 Probability of Becoming Unmatched

Consider child (x,q) at the beginning of a period, and let $U(x,q)$ denote the probability that she becomes unmatched:

$$U(x,q) = (1-\delta_x)\big(1-a(x,q)\big)\Big\{ \underbrace{d(x,q)\Big[1-\pi^c(\theta)\sum_{\mathcal{M}(x,q_0)} f(s)\sum_{q'} g(q'|s)\big(1-d(x,q')\big)\Big]}_{1-M(x)} + \big(1-d(x,q)\big)\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\sum_{q'} g(q'|s)d(x,q')\Big\}$$

Here, child (x,q) becomes unmatched if f-match (x,q) is destroyed and she remains unmatched after the search and f-matching stage, or if the child in the f-match (x,q) forms a new f-match which is later on destroyed.

Corollary 4. *In any foster care equilibrium, the probability of becoming unmatched is:*

(*i*) *greater for children* ***with a disability*** *if* $\frac{\delta_{x_2}-\delta_{x_1}}{1-\delta_{x_1}} \geq f(s_1)$ *and* $\frac{1-\delta_{x_1}}{2-\delta_{x_1}-\delta_{x_2}} > \overline{\pi}$ *hold.*

(*ii*) *greater for* ***low-quality*** *matches if* $a(x,q_1)=0$ *and* $a(x,q_2)=0$ *hold.*

Proof. See Appendix E.4. □

Corollary 4(i) exhibits sufficient conditions for Fact 4 to arise in equilibrium. I say that disability increases the probability of becoming unmatched if $U(x_1,q) \geq U(x_2,q)$ for all q. There are potentially two driving forces working on opposite directions in this case. On the one hand, by Proposition 1(i) and Corollary 3, children with a disability are more likely to destroy an f-match and more likely to remain unmatched, which makes them more likely to become unmatched. On the other hand, by Propositions 1(i) and 3(i), children with a disability are less likely to form a new f-match but are more likely to destroy the new f-match later on, thus is not clear who is more likely to become unmatched.

Corollary 4(ii) shows that the probability of becoming unmatched is decreasing in match quality, $U(x,q_1) \geq U(x,q_2)$ for all x. In this case, the driving forces behind becoming unmatched are aligned.

5 Concluding Remarks

This paper provides an extensive analysis of the match transitions of children relinquished for adoption in the US foster care system. I first present an empirical analysis that yields four new facts. Thereafter, I develop a two-sided search and matching model used to rationalize the empirical facts and carry out predictions regarding match quality.

Using the theoretical model, I show that foster match separation involving children with a disability is mainly driven by the uncertainty of the quality of the match, while foster separation involving children without a disability is driven to improve match quality. Also, I find that high-quality matches are less likely to be separated. Surprisingly, I find that foster match separation plays a crucial role in adoption by influencing the incentives of foster parents to adopt. Due to the presence of the financial penalty on adoption, parents face the following trade-off when deciding to adopt: accept the penalty in exchange for eliminating the likelihood that the child breaks the foster match in the future. For adoption, I show that the adoption penalty not only exacerbates the intrinsic disadvantage faced by children with a disability but also creates incentives for high-quality matches to not transit to adoption. Moreover, I show that foster parents in high-quality matches might have fewer incentives to adopt.

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A Appendix: Tables

Table A1: Descriptive Statistics by Disability Status

	Sample A		Sample B		Sample C	
	$obs = 1,165,818$		$obs = 659,253$		$obs = 65,970$	
Disability	Yes	No	Yes	No	Yes	No
Adopted	0.22	0.32	-	-	-	-
	(0.41)	(0.47)				
Foster matched	0.89	0.96	1.00	1.00	0.00	0.00
	(0.31)	(0.19)	(0.00)	(0.00)	(0.00)	(0.00)
Becomes foster matched	-	-	-	-	0.22	0.28
					(0.41)	(0.45)
Becomes unmatched	-	-	0.03	0.01	-	-
			(0.18)	(0.11)		
Foster match separates	-	-	0.19	0.18	-	-
			(0.40)	(0.38)		
Age in years	7.96	6.01	7.79	6.07	12.34	11.81
	(4.45)	(4.23)	(4.35)	(4.22)	(2.56)	(3.22)
Male	0.57	0.50	0.60	0.50	0.65	0.58
	(0.50)	(0.50)	(0.50)	(0.50)	(0.48)	(0.49)
White	0.43	0.44	0.41	0.42	0.45	0.42
	(0.49)	(0.50)	(0.49)	(0.50)	(0.50)	(0.49)
Black	0.25	0.23	0.27	0.24	0.26	0.29
	(0.43)	(0.42)	(0.44)	(0.43)	(0.44)	(0.45)
Hispanic	0.22	0.22	0.22	0.23	0.19	0.21
	(0.41)	(0.41)	(0.41)	(0.42)	(0.39)	(0.41)
Title IV-E eligible	0.49	0.47	0.53	0.50	0.45	0.50
	(0.50)	(0.50)	(0.50)	(0.50)	(0.50)	(0.50)
Months in foster care	41.20	30.53	40.91	30.23	55.94	46.99
	(28.66)	(19.80)	(28.62)	(20.37)	(37.34)	(34.54)
Months since PRT*	21.71	13.91	20.09	13.29	44.14	37.34
	(26.53)	(18.83)	(25.27)	(18.92)	(37.02)	(36.09)
Months in current placement	16.67	15.64	18.18	16.64	11.03	10.47
	(17.32)	(14.51)	(18.16)	(14.72)	(14.67)	(12.38)

Notes: Data are from Adoption and Foster Care Analysis and Reporting System (AFCARS). Means and standard deviations are calculated for child-period observations. Sample A is the full sample containing all children younger than age 16 whose parental rights have been terminated and who are either foster matched or unmatched. Sample B and Sample C are subsamples of A. Sample B (sample C) keeps only those child-period observations such that the child is foster matched (unmatched) at the beginning of the period and still in foster care at the end of the period.
* PRT stands for Parental Rights Terminated.

Table A2: Stylized Facts from Foster Care - Effect of Disability

	Foster matched
Disability γ	-0.043***
	(0.002)
Mean of dependent variable	0.934
Number of child-period observations	1,165,818

Notes: Data are from Adoption and Foster Care Analysis and Reporting System (AFCARS). All specifications control for child's demographics, states indicators and period indicators. The first and second columns consider sample A, third and fifth columns use sample B, and the fourth column uses sample C. Standard errors are cluster at the state-period level and shown in parentheses. ***$P < 0.01$; **$P < 0.05$; *$P < 0.10$.

B Appendix: Omitted Equations

B.1 Endogenous Distribution of Children

A parent can meet a child who is unmatched or f-matched with quality q. Thus, an unmatched parent meets a child $(x, \bar{q})$ according to the probability distribution $\hat{m}(x, \bar{q})$ where:

$$\hat{m}(x,\bar{q}) = \begin{cases} \frac{u^c(x)}{\sum_x u^c(x)+\sum_q m(x,q)} & \text{if } \bar{q} = q_0 \\ \frac{m(x,q)}{\sum_x u^c(x)+\sum_q m(x,q)} & \text{if } \bar{q} = q \end{cases} \tag{B.1}$$

Therefore, a parent meets an unmatched child x with total probability $\pi^p(\theta)\hat{m}(x, q_0)$. Similarly, a parent meets a child (x, q) with total probability $\pi^p(\theta)\hat{m}(x, q)$.

B.2 Aggregate State of the Market

For each (x, q), $m(x, q)$ satisfies the following equality:

$$\underbrace{m(x,q)\Big\{\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s) + \Big(1-\pi^c(\theta_t)\sum_{\mathcal{M}(x,q)} f(s)\Big)\Big[\delta_x + \big(1-\delta_x\big)d(x,q)a(x,q)\Big]\Big\}}_{mass\ destruction} =$$

$$\underbrace{\begin{aligned} &u^c(x)\pi^c(\theta)\sum_{\mathcal{M}(x,q_0)} f(s)g(q|s)\big(1-\delta_x\big)\Big(1-d(x,q)\Big)\Big(1-a(x,q)\Big) \\ &+\sum_{q'} m(x,q')\pi^c(\theta)\sum_{\mathcal{M}(x,q')} f(s)g(q|s)\big(1-\delta_x\big)\Big(1-d(x,q)\Big)\Big(1-a(x,q)\Big)\end{aligned}}_{mass\ creation} \tag{B.2}$$

For each x, $u^c(x)$ satisfies the following equality:

$$\begin{aligned} u^c(x)\Big\{\pi^c(\theta)\sum_{\mathcal{M}(x,q_0)} f(s)\sum_q\Big[\delta_x + \big(1-\delta_x\big)\big(1-d(x,q)\big)g(q|s)\Big] \\ + \Big(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q_0)} f(s)\Big)\delta_x\Big\} = \\ \sum_q m(x,q)\big(1-\delta_x\big)\Big\{\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\sum_{q'} g(q'|s)d(x,q') + \Big(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\Big)d(x,q)\Big\} \\ + \rho l(x) \end{aligned} \tag{B.3}$$

B.3 Children' Decision Conditions

Child (x,q) chooses $a^c(x,q)=1$ if and only if:

$$\frac{b^c(x,q,a)}{1-\beta} > \max\left\{ \Big(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q_0)} f(s)\Big)\mathcal{C}(x,q_0) + \pi^c(\theta)\sum_{\mathcal{M}(x,q_0)} \mathbb{E}_s[\mathcal{C}(x,q)]f(s)\ , \right.$$
$$\left. \Big(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\Big)\mathcal{C}(x,q) + \pi^c(\theta)\sum_{\mathcal{M}(x,q)} \mathbb{E}_s[\mathcal{C}(x,q)]f(s) \right\} \tag{B.4}$$

Child (x,q) chooses $d^c(x,q)=1$ if and only if:

$$\Big(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q_0)} f(s)\Big)\mathcal{C}(x,q_0) + \pi^c(\theta)\sum_{\mathcal{M}(x,q_0)} \mathbb{E}_s[\mathcal{C}(x,q)]f(s) > \max\left\{ \frac{b^c(x,q,a)}{1-\beta}\ , \right.$$
$$\left. \Big(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\Big)\mathcal{C}(x,q) + \pi^c(\theta)\sum_{\mathcal{M}(x,q)} \mathbb{E}_s[\mathcal{C}(x,q)]f(s) \right\} \tag{B.5}$$

B.4 Parents' Decision Conditions

A parent chooses $in=1$ if and only if:

$$-k + \beta\pi^p(\theta)\sum_{\mathcal{M}(x,\bar{q})}\sum_{x,\bar{q}} \mathbb{E}_s[\mathcal{P}(x,q)]\hat{m}(x,\bar{q})f(s) > 0 \tag{B.6}$$

Parent (x,q) chooses $a^p(x,q)=1$ if and only if:

$$\frac{b^p(x,q,a)}{1-\beta} > \max\left\{ \Big(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\Big)\cdot \frac{b^p(x,q,f)}{1-\beta(1-\delta_x)\big(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\big)}\ , \mathcal{P}^u \right\} \tag{B.7}$$

Parent (x,q) chooses $d^p(x,q)=1$ if and only if:

$$\mathcal{P}^u > \max\left\{ \Big(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\Big)\cdot \frac{b^p(x,q,f)}{1-\beta(1-\delta_x)\big(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\big)}\ , \frac{b^p(x,q,a)}{1-\beta} \right\} \tag{B.8}$$

C Appendix: Definition of Equilibrium

I use the following equilibrium definition:

Definition 2. *A* ***foster care equilibrium*** *consists of tuple* $(\mathcal{M}, d^c, d^p, a^c, a^p, in, \mathcal{C}, \mathcal{P}^u, \mathcal{P}, \phi)$ *such that the following properties are satisfied:*

(1) ***Value Functions.***

(a) *Given* $(\mathcal{M}, d^c, d^p, a^c, a^p, \phi)$*, value functions* $\mathcal{C}(x, q_0)$ *and* $\mathcal{C}(x, q)$ *are specified by Equations 3 and 4, respectively.*

(b) *Given* $(\mathcal{M}, d^c, d^p, a^c, a^p, in, \phi)$*, value functions* $\mathcal{P}^u$ *and* $\mathcal{P}(x, q)$ *are specified by Equations 5 and 6, respectively.*

(2) ***Strategies.***

(a) *Given* $(d^c, d^p, a^c, a^p, \mathcal{C}, \mathcal{P}^u, \mathcal{P}, \phi)$*,* $s \in \mathcal{M}(x, \bar{q})$ *if and only if* $\mathbb{E}_s[\mathcal{P}(x, q)] \geq \mathcal{P}^u$ *and* $\mathbb{E}_s[\mathcal{C}(x, q)] \geq \mathcal{C}(x, \bar{q})$*.*

(b) *Given* $(\mathcal{M}, d^p, a^p, \mathcal{C}, \phi)$*,* $a^c(x, q) = 1$ *if and only if Equation B.4 holds, and* $d^c(x, q) = 1$ *if and only if Equation B.5 holds.*

(c) *Given* $(\mathcal{M}, d^c, a^c, \mathcal{P}^u, \mathcal{P}, \phi)$*,* $in = yes$ *if and only if B.6 holds,* $a^p(x, q) = 1$ *if and only if Equation B.7 holds, and* $d^p(x, q)$ *is one if and only if Equation B.8 holds.*

(3) ***Aggregate state of the market.***

(a) *Given* $(\mathcal{M}, d^c, d^p, a^c, a^p, in, \mathcal{P}^u, \mathcal{P}, u^c, m)$*,* u^p *satisfies Equation 7.*

(b) *Given* $(\mathcal{M}, d^c, d^p, a^c, a^p)$*, for each* x*,* $\{m(x, q_i)\}_{i=1}^N$ *and* $u^c(x)$ *solve the system of equations given by Equations B.2 and B.3.*

D Appendix: Proofs of Equilibrium Analysis

D.1 Proof of Proposition 1

I start by describing the destruction strategies of children and parents. Lemma 1 states that, in any foster care equilibrium, child (x, q) does not destroy if $b^c(x, q, f)$ is non-negative.

Lemma 1 (**Destruction Strategies of Children**)**.** *In any foster care equilibrium, $d^c(x, q) = 0$ if $b^c(x, q, f) \geq 0$ for all (x, q).*

Proof. Fixing (x, q), assume that $b^c(x, q, f)$ is non-negative. By contradiction, suppose $d^c(x, q) = 1$ then, by the equilibrium definition, it follows that $\hat{\mathcal{C}}(x, q_0) > \hat{\mathcal{C}}(x, q)$, that is:

$$\Big(1 - \pi^c(\theta) \sum_{\mathcal{M}(x,q_0)} f(s)\Big)\mathcal{C}(x, q_0) + \pi^c(\theta) \sum_{\mathcal{M}(x,q_0)} \mathbb{E}_s[\mathcal{C}(x, q)]f(s) > \Big(1 - \pi^c(\theta) \sum_{\mathcal{M}(x,q)} f(s)\Big)\mathcal{C}(x, q) + \pi^c(\theta) \sum_{\mathcal{M}(x,q)} \mathbb{E}_s[\mathcal{C}(x, q)]f(s) \quad \text{(D.1)}$$

By assumption $\hat{\mathcal{C}}(x, q_0) > \hat{\mathcal{C}}(x, q)$, then the value function for child (x, q) is:

$$\begin{aligned}\mathcal{C}(x, q) &= b^c(x, q, f) + \beta\delta_x \frac{b^c(x, q_2, a)}{1 - \beta} \\ &\quad + \beta(1 - \delta_x)\Big[\Big(1 - \pi^c(\theta) \sum_{\mathcal{M}(x,q_0)} f(s)\Big)\mathcal{C}(x, q_0) + \pi^c(\theta) \sum_{\mathcal{M}(x,q_0)} \mathbb{E}_s[\mathcal{C}(x, q)]f(s)\Big]\end{aligned}$$

Since $b^c(x, q, f)$ is non-negative, it follows that:

$$\begin{aligned}\mathcal{C}(x, q) &= b^c(x, q, f) + \beta\delta_x \frac{b^c(x, q_2, a)}{1 - \beta} \\ &\quad + \beta(1 - \delta_x)\Big[\Big(1 - \pi^c(\theta) \sum_{\mathcal{M}(x,q_0)} f(s)\Big)\mathcal{C}(x, q_0) + \pi^c(\theta) \sum_{\mathcal{M}(x,q_0)} \mathbb{E}_s[\mathcal{C}(x, q)]f(s)\Big] \\ &\geq \beta\delta_x \frac{b^c(x, q_2, a)}{1 - \beta} \\ &\quad + \beta(1 - \delta_x)\Big[\Big(1 - \pi^c(\theta) \sum_{\mathcal{M}(x,q_0)} f(s)\Big)\, \mathcal{C}(x, q_0) + \pi^c(\theta) \sum_{\mathcal{M}(x,q_0)} \mathbb{E}_s[\mathcal{C}(x, q)]f(s)\Big] \\ &= \mathcal{C}(x, q_0)\end{aligned}$$

In equilibrium, $s \in \mathcal{M}(x, \bar{q})$ if and only if $\mathbb{E}_s[\mathcal{C}(x, q)] \geq \mathcal{C}(x, \bar{q})$ and $\mathbb{E}_s[\mathcal{P}(x, q)] \geq \mathcal{P}^u$. Thus, if $\mathcal{C}(x, q) \geq \mathcal{C}(x, q_0)$ then $\mathcal{M}(x, q) \subseteq \mathcal{M}(x, q_0)$. Now, I show that $\mathcal{C}(x, q) \geq \mathcal{C}(x, q_0)$ contradicts $\hat{\mathcal{C}}(x, q_0) > \hat{\mathcal{C}}(x, q)$. For this, I analyze two cases:

Case 1: Suppose $\mathcal{C}(x,q) = \mathcal{C}(x,q_0)$, then $\mathcal{M}(x,q) = \mathcal{M}(x,q_0)$. Thus, $\hat{\mathcal{C}}(x,q) = \hat{\mathcal{C}}(x,q_0)$ which implies that $d^c(x,q) = 0$. A contradiction.

Case 2: Suppose $\mathcal{C}(x,q) > \mathcal{C}(x,q_0)$, then $\mathcal{M}(x,q) \subset \mathcal{M}(x,q_0)$. Here, I define the set $\hat{\mathcal{M}}(x,q) = \{s \in S | s \in \mathcal{M}(x,q_0) \setminus \mathcal{M}(x,q)\}$. Thus, the following holds:

$$\begin{aligned}
\hat{\mathcal{C}}(x,q) =& \Big(1 - \pi^c(\theta) \sum_{\mathcal{M}(x,q)} f(s)\Big)\mathcal{C}(x,q) + \pi^c(\theta) \sum_{\mathcal{M}(x,q)} \mathbb{E}_s[\mathcal{C}(x,q)]f(s) \\
=& \Big(1 - \pi^c(\theta)\Big)\mathcal{C}(x,q) + \pi^c(\theta) \sum_{s \notin \mathcal{M}(x,q)} f(s)\mathcal{C}(x,q) + \pi^c(\theta) \sum_{\mathcal{M}(x,q)} \mathbb{E}_s[\mathcal{C}(x,q)]f(s) \\
=& \Big(1 - \pi^c(\theta)\Big)\mathcal{C}(x,q) + \pi^c(\theta) \sum_{s \notin \mathcal{M}(x,q_0)} f(s)\mathcal{C}(x,q) + \pi^c(\theta) \sum_{s \in \hat{\mathcal{M}}(x,q)} f(s)\mathcal{C}(x,q) \\
& + \pi^c(\theta) \sum_{\mathcal{M}(x,q)} \mathbb{E}_s[\mathcal{C}(x,q)]f(s) \\
>& \Big(1 - \pi^c(\theta)\Big)\mathcal{C}(x,q_0) + \pi^c(\theta) \sum_{s \notin \mathcal{M}(x,q_0)} f(s)\mathcal{C}(x,q_0) + \pi^c(\theta) \sum_{s \in \hat{\mathcal{M}}(x,q)} f(s)\mathcal{C}(x,q) \\
& + \pi^c(\theta) \sum_{\mathcal{M}(x,q)} \mathbb{E}_s[\mathcal{C}(x,q)]f(s)
\end{aligned}$$

By definition, if $s \in \hat{\mathcal{M}}(x,q)$ then $\mathcal{C}(x,q) > \mathbb{E}_s[\mathcal{C}(x,q)] > \mathcal{C}(x,q_0)$. Thus, the following holds:

$$\begin{aligned}
\hat{\mathcal{C}}(x,q) >& \Big(1 - \pi^c(\theta)\Big)\mathcal{C}(x,q_0) + \pi^c(\theta) \sum_{s \notin \mathcal{M}(x,q_0)} f(s)\mathcal{C}(x,q_0) + \pi^c(\theta) \sum_{s \in \hat{\mathcal{M}}(x,q)} f(s)\mathcal{C}(x,q) \\
& + \pi^c(\theta) \sum_{\mathcal{M}(x,q)} \mathbb{E}_s[\mathcal{C}(x,q)]f(s) \\
>& \Big(1 - \pi^c(\theta) \sum_{\mathcal{M}(x,q_0)} f(s)\Big)\mathcal{C}(x,q_0) + \pi^c(\theta) \sum_{\mathcal{M}(x,q_0)} \mathbb{E}_s[\mathcal{C}(x,q)]f(s) = \hat{\mathcal{C}}(x,q_0)
\end{aligned}$$

which contradicts equation D.1. Hence, if $b^c(x,q,f) \geq 0$ then $d^c(x,q) = 0$. □

Lemma 2 shows that parents destroy an f-match of quality q with child x if and only if $b^p(x,q,f)$ is negative.

Lemma 2 (**Destruction Strategies of Parents**). *Assume parents' payoffs satisfy Assumption 2(a). In any foster care equilibrium, $d^p(x,q) = 1$ if and only if $b^p(x,q,f) < 0$ for all (x,q).*

Proof. (⇒) Fix (x,q). Assume $d^p(x,q) = 1$, then the following inequality must hold:

$$0 > \max\left\{\left(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\right)\cdot\frac{b^p(x,q,f)}{1-\beta(1-\delta_x)\left(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\right)}, \frac{b^p(x,q,a)}{1-\beta}\right\}$$

By contradiction, suppose $b^p(x,q,f)$ is non-negative. Since $1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s) \geq 0$ and $1-\beta(1-\delta_x)\left(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\right) \geq 0$, there is a contradiction. Hence, $d^p(x,q)=1$ only if $b^p(x,q,f)$ is negative.

($\Leftarrow$) Fixing (x,q), assume that $b^p(x,q,f)$ is negative. By contradiction, suppose $d^p(x,q)=0$. There are two possible cases:

Case 1: Suppose $a^p(x,q)=1$, then $\frac{b^p(x,q,a)}{1-\beta} > 0$ must hold. Since $b^p(x,q,f)$ is negative then, by assumption 2(a), $b^p(x,q,a)$ is also negative. Hence, there is a contradiction.

Case 2: Suppose $a^p(x,q)=0$ and $d^p(x,q)=0$, then the following inequality must hold:

$$\left(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\right)\cdot\frac{b^p(x,q,f)}{1-\beta(1-\delta_x)\left(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\right)} \geq \max\left\{0, \frac{b^p(x,q,a)}{1-\beta}\right\}$$

Since $b^p(x,q,f)$ is negative then, by assumption 2(a), $b^p(x,q,a)$ is also negative. Thus, it must be that $d^p(x,q)=1$. □

Now, I prove Proposition 1 using Lemmas 1 and 2. By Lemma 1 and Assumption 1(a), it follows that $d^c(x,q)=0$ for all (x,q). This implies that the total probability of destruction of an f-match depends on the destruction strategies of parents.

(i) Fix some quality q. Suppose a parent f-matched to child x_2 when the quality is q chooses $d^p(x_2,q)=1$. Then, by Lemma 2, $b^p(x_2,q,f)$ is negative. By Assumption 2(b), $b^p(x_1,q,f)$ is also negative. Thus, by Lemma 2, $d^p(x_1,q)=1$. Hence, $d(x_1,q)\geq d(x_2,q)$.

(ii) Fix some child x. Suppose a parent f-matched to child x when the quality is q chooses $d^p(x,q)=1$. Then, by Lemma 2, $b^p(x,q,f)$ is negative. Now, consider q' such that $q>q'$. By Assumption 2(c), $b^p(x,q',f)$ is also negative. Thus, by Lemma 2, $d^p(x,q')=1$. Hence, $d(x,q')\geq d(x,q)$ whenever $q>q'$.

D.2 Proof of Proposition 2

I start by describing the f-match formation strategies of unmatched children and parents. Lemma 3 shows that child (x,q_0) announces foster after observing signal s if $\mathbb{E}_s[b^c(x,q,f)]$ is non-negative.

Lemma 3 (**F-match Formation Strategies of Unmatched Children**). *In any foster care equilibrium, $s \in F^c(x, q_0)$ if $\mathbb{E}_s[b^c(x, q, f)] \geq 0$ for all (x, s).*

Proof. Fix x. In any foster care equilibrium, $s \in F^c(x, q_0)$ if and only if $\mathbb{E}_s[\mathcal{C}(x, q)] \geq \mathcal{C}(x, q_0)$. I show that, for all $s \in S$, if $\mathbb{E}_s[b^c(x, q, f)] \geq 0$ then $\mathbb{E}_s[\mathcal{C}(x, q)] \geq \mathcal{C}(x, q_0)$. Note that, since the destruction of f-matches is unilateral, the conditional expected value $\mathbb{E}_s[\mathcal{C}(x, q)]$ is bounded bellow by $\sum_q b^c(x, q, f)g(q|s) + \beta\delta_x \frac{b^c(x,q_2,a)}{1-\beta} + \beta(1 - \delta_x)\hat{\mathcal{C}}(x, q_0)$. Assuming that $\mathbb{E}_s[b^c(x, q, f)]$ is non-negative, the following inequality holds:

$$\begin{aligned}
\mathbb{E}_s[\mathcal{C}(x, q)] &\geq \sum_q b^c(x, q, f)g(q|s) + \beta\delta_x \frac{b^c(x, q_2, a)}{1 - \beta} + \beta(1 - \delta_x)\hat{\mathcal{C}}(x, q_0) \\
&\geq \beta\delta_x \frac{b^c(x, q_2, a)}{1 - \beta} + \beta(1 - \delta_x)\hat{\mathcal{C}}(x, q_0) = \mathcal{C}(x, q_0)
\end{aligned}$$

Hence, if $\mathbb{E}_s[b^c(x, q, f)] \geq 0$ then $\mathbb{E}_s[\mathcal{C}(x, q)] \geq \mathcal{C}(x, q_0)$. □

Lemma 4 establishes that parents announces foster after observing signal s if and only if the conditional expected payoff of being f-matched is non-negative.

Lemma 4 (**F-match Formation Strategies of Parents**). *Assume parents' payoffs satisfy Assumption 4. In any foster care equilibrium, $s \in F^p(x)$ if and only if $\mathbb{E}_s[b^p(x, q, f)] \geq 0$ for all (x, s).*

Proof. ($\Rightarrow$) Fix x. In any foster care equilibrium, $s \in F^p(x)$ if and only if $\mathbb{E}_s[\mathcal{P}(x, q)] \geq 0$. I show that if $\mathbb{E}_s[b^p(x, q, f)] \geq 0$ then $\mathbb{E}_s[\mathcal{P}(x, q)] \geq 0$. Fixing s, consider the conditional expected value:

$$\begin{aligned}
&\mathbb{E}_s[\mathcal{P}(x, q)] = \sum_q b^p(x, q, f)g(q|s) \\
&+ \beta(1 - \delta_x) \sum_q \left[a^c(x, q) \max \left\{ \frac{b^p(x, q, a)}{1 - \beta}, 0, \frac{\left(1 - \pi^c(\theta) \sum_{\mathcal{M}(x,q)} f(s)\right) b^p(x, q, f)}{1 - \beta(1 - \delta_x)\left(1 - \pi^c(\theta) \sum_{\mathcal{M}(x,q)} f(s)\right)} \right\} \right. \\
&\left. + \left(1 - d^p(x, q) - a^p(x, q)\right) \max \left\{ 0, \frac{\left(1 - \pi^c(\theta) \sum_{\mathcal{M}(x,q)} f(s)\right) b^p(x, q, f)}{1 - \beta(1 - \delta_x)\left(1 - \pi^c(\theta) \sum_{\mathcal{M}(x,q)} f(s)\right)} \right\} \right] g(q|s)
\end{aligned}$$

Since f-match destruction is unilateral, the conditional expected value $\mathbb{E}_s[\mathcal{P}(x, q)]$ is bounded bellow by $\mathbb{E}_s[b^p(x, q, f)]$.Thus, if $\mathbb{E}_s[b^p(x, q, f)] \geq 0$ then $\mathbb{E}_s[\mathcal{P}(x, q)] \geq 0$.

($\Leftarrow$) Fix (x, s). I show that, if $\mathbb{E}_s[b^p(x, q, f)]$ is negative then $\mathbb{E}_s[\mathcal{P}(x, q)]$ is also negative. Note that, $\mathbb{E}_s[\mathcal{P}(x,q)]$ is bounded above by the following expression:

$$\sum_q \overline{\mathcal{P}(x,q)g(q|s)} = \sum_q b^p(x,q,f)g(q|s) + \beta(1-\delta_x)\sum_q \left[\max\left\{\frac{b^p(x,q,a)}{1-\beta}\,,0\,,\frac{b^p(x,q,f)}{1-\beta\,(1-\delta_x)}\right\}\right]g(q|s)$$

Since $\sum_q b^p(x,q)g(q|s)$ is negative, by Assumption 4, $\sum_q \overline{\mathcal{P}(x,q)g(q|s)}$ is also negative. □

Now, I prove Proposition 2 using Lemmas 3 and 4. By definition, f-matches must be mutually agreed upon $s \in \mathcal{M}(x,q_0)$ if and only if $s \in F^c(x,q_0)$ and $s \in F^p(x)$. By Assumption 1(a), it follows that $\mathbb{E}_s[b^c(x,q,f)] \geq 0$ for all $s \in S$. Hence, by Lemma 3, $F^c(x,q_0) = S$.

(i) Fix signal s, I show that if $s \in F^p(x_1)$ then $s \in F^p(x_2)$. Suppose $s \in F^p(x_1)$ then, by Lemma 4, it follows that $\mathbb{E}_s[b^p(x_1,q,f)]$ must be non-negative. Since $b^p(x_2,q,f) \geq b^p(x_1,q,f)$ for all q, by Assumption 2(b), then $\mathbb{E}_s[b^p(x_2,q,f)]$ is also non-negative. Thus, by Lemma 4, $s \in F^p(x_2)$. By Assumption 3, it follows that $F^p(x_1)$ and $F^p(x_2)$ are non-empty. Hence, $\mathcal{M}(x,q_0)$ is non-empty for all x, and $\mathcal{M}(x_1,q_0) \subseteq \mathcal{M}(x_2,q_0)$.

(ii) Fix child x. Consider s and s' such that $s' > s$. I show that, if $s \in F^p(x)$ then $s' \in F^p(x)$. Suppose $s \in F^p(x)$ then, by Lemma 4, it follows that $\mathbb{E}_s[b^p(x,q,f)]$ is non-negative. Given that $G(q|s') \leq G(q|s)$ and $b^p(x,q,f)$ is increasing in q (Assumption 2(c)), it follows that $\mathbb{E}_{s'}[b^p(x,q,f)]$ is also non-negative. Hence, by Lemma 4, $s' \in F^p(x)$. Hence, if $s \in \mathcal{M}(x,q_0)$ then $s' \in \mathcal{M}(x,q_0)$.

D.3 Proof of Proposition 3

First I establish that, as a best-response, children with and without a disability choose the same f-match formation strategy, and both are more willing to separate from an f-match of low-quality q_1 than a high-quality match q_2.

Lemma 5 (**F-match formation strategies of f-matched children**)**.** *Assume children' payoffs satisfy Assumptions 1(a),(c)-(d). Then, for all x, $F^c(x,q_1) = S$ and $F^c(x,q_2) = \{\emptyset\}$ whenever $d(x,q_1) \geq d(x,q_2)$ holds.*

Proof. For each x, it follows that $F^c(x,q_2) \cap F^c(x,q_1) = S$ or $F^c(x,q_2) \cap F^c(x,q_1) = \{\emptyset\}$. The reason is the following. For each signal, $s \in F^c(x,q_2)$ if and only if $\mathbb{E}_s[\mathcal{C}(x,q)] = \mathcal{C}(x,q_1)g(q_1|s) + \mathcal{C}(x,q_2)g(q_2|s) \geq \mathcal{C}(x,q_2)$. Then, it must be that

$\mathcal{C}(x, q_1) \geq \mathcal{C}(x, q_2)$ independent of the distributions. Now, if $\mathcal{C}(x, q_1) > \mathcal{C}(x, q_2)$ then $s \notin F^c(x, q_1)$, and if $\mathcal{C}(x, q_1) = \mathcal{C}(x, q_2)$ then $s \in F^c(x, q_1)$. Hence, there are three possible cases:

(1) $F^c(x, q_2) = F^c(x, q_1) = \{s_1, s_2\}$.

(2) $F^c(x, q_2) = \{s_1, s_2\}$ and $F^c(x, q_1) = \{\emptyset\}$.

(3) $F^c(x, q_2) = \{\emptyset\}$ and $F^c(x, q_1) = \{s_1, s_2\}$.

I show that $\mathcal{C}(x, q_2) > \mathcal{C}(x, q_1)$ holds, thus only the third case is feasible. Since $d(x, q_1) \geq d(x, q_2)$, the following cases might arise:

<u>Case a:</u> Suppose $a(x, q_1) = 1$, then $\mathcal{C}(x, q_1) = b^c(x, q_1, f) + \beta\delta_x \frac{b^c(x,q_2,a)}{1-\beta} + \beta(1 - \delta_x)\frac{b^c(x,q_1,a)}{1-\beta}$

(a1) If $a(x, q_2) = 1$, then $\mathcal{C}(x, q_2) - \mathcal{C}(x, q_1) = b^c(x, q_2, f) - b^c(x, q_1, f) + \beta(1 - \delta_x)\left[\frac{b^c(x,q_2,a)}{1-\beta} - \frac{b^c(x,q_1,a)}{1-\beta}\right]$

(a2) If $a(x, q_2) = 0$ and $d(x, q_2) = 0$, then $\mathcal{C}(x, q_2) - \mathcal{C}(x, q_1) = b^c(x, q_2, f) - b^c(x, q_1, f) + \beta(1 - \delta_x)\left[\hat{\mathcal{C}}(x, q_2) - \frac{b^c(x,q_1,a)}{1-\beta}\right]$

<u>Case b:</u> Suppose $d(x, q_1) = 1$, then $\mathcal{C}(x, q_1) = b^c(x, q_1, f) + \beta\delta_x \frac{b^c(x,q_2,a)}{1-\beta} + \beta(1 - \delta_x)\hat{\mathcal{C}}(x, q_0)$

(b1) If $d(x, q_2) = 1$, then $\mathcal{C}(x, q_2) - \mathcal{C}(x, q_1) = b^c(x, q_2, f) - b^c(x, q_1, f)$

(b2) If $a(x, q_2) = 1$, then $\mathcal{C}(x, q_2) - \mathcal{C}(x, q_1) = b^c(x, q_2, f) - b^c(x, q_1, f) + \beta(1 - \delta_x)\left[\frac{b^c(x,q_2,a)}{1-\beta} - \hat{\mathcal{C}}(x, q_0)\right]$

(b3) If $a(x, q_2) = 0$ and $d(x, q_2) = 0$, then $\mathcal{C}(x, q_2) - \mathcal{C}(x, q_1) = b^c(x, q_2, f) - b^c(x, q_1, f) + \beta(1 - \delta_x)\left[\hat{\mathcal{C}}(x, q_2) - \hat{\mathcal{C}}(x, q_0)\right]$

<u>Case c:</u> Suppose $a(x, q_1) = 0$ and $d(x, q_1) = 0$, then $\mathcal{C}(x, q_1) = b^c(x, q_1, f) + \beta\delta_x \frac{b^c(x,q_2,a)}{1-\beta} + \beta(1 - \delta_x)\hat{\mathcal{C}}(x, q_1)$

(c1) If $a(x, q_2) = 1$, then $\mathcal{C}(x, q_2) - \mathcal{C}(x, q_1) = b^c(x, q_2, f) - b^c(x, q_1, f) + \beta(1 - \delta_x)\left[\frac{b^c(x,q_2,a)}{1-\beta} - \hat{\mathcal{C}}(x, q_1)\right]$

(c2) If $a(x, q_2) = 0$ and $d(x, q_2) = 0$, then $\mathcal{C}(x, q_2) - \mathcal{C}(x, q_1) = b^c(x, q_2, f) - b^c(x, q_1, f) + \beta(1 - \delta_x)\left[\hat{\mathcal{C}}(x, q_2) - \hat{\mathcal{C}}(x, q_1)\right]$

Assume 1(a) and 1(c), then $\mathcal{C}(x, q_2) - \mathcal{C}(x, q_1) > 0$ for cases (a1) and (b1). For case (b3), if $d(x, q_2) = 0$ then $\hat{\mathcal{C}}(x, q_2) \geq \hat{\mathcal{C}}(x, q_0)$. Thus, by Assumptions 1(a) and 1(c), it follows that $\mathcal{C}(x, q_2) - \mathcal{C}(x, q_1) > 0$ in case (b3). By assumption 1(d), it follows that $\frac{b^c(x,q_2,a)}{1-\beta} \geq \hat{\mathcal{C}}(x, \bar{q})$ for all q. Hence, by assumptions 1(a),(c)(d) it follows that $\mathcal{C}(x, q_2) - \mathcal{C}(x, q_1) > 0$ for all the other cases. Therefore, $F^c(x_1, q_2) = F^c(x_2, q_2) = \{\emptyset\}$ and $F^c(x_1, q_1) = F^c(x_2, q_1) = \{s_1, s_2\}$. □

Now, I establish Proposition 3. By Assumptions 1(a), 2(a), and 2(c), Proposition 1(ii) holds. Thus, for children, Lemma 5 holds. For parents, by Assumptions 2(b), 3, and 4, Proposition 2(i) holds, that is, $F^p(x)$ is non-empty for all x, and $F^p(x_1) \subseteq F^p(x_2)$. Moreover, by adding Assumption 2(c), Proposition 2(ii) holds. That is, for all x, if $s_1 \in F^p(x)$ then $s_2 \in F^p(x)$.

Since $s \in \mathcal{M}(x, q)$ if and only if $s \in F^c(x, q)$ and $s \in F^p(x)$, the following holds:

(a) $\mathcal{M}(x, q_1)$ is non-empty for all x.

(b) $\mathcal{M}(x, q_2) = \{\emptyset\}$ for all x.

(c) $\mathcal{M}(x_1, q_1) \subseteq \mathcal{M}(x_2, q_1)$.

(d) $s_1 \in \mathcal{M}(x, q_1)$ implies $s_2 \in \mathcal{M}(x, q_1)$ for all x.

Finally, since $\mathcal{M}(x, q_2) = \{\emptyset\}$, then $\sum_{\mathcal{M}(x,q_2)} f(s) = 0$. Hence, $\sum_{\mathcal{M}(x,q_1)} f(s) \geq \sum_{\mathcal{M}(x,q_2)} f(s)$. Now, since $\mathcal{M}(x_1, q_1) \subseteq \mathcal{M}(x_2, q_1)$ then $\sum_{\mathcal{M}(x_2,q)} f(s) \geq \sum_{\mathcal{M}(x_1,q)} f(s)$ for all q.

D.4 Proof of Proposition 4

I start by describing the adoption strategies of children and parents. Lemma 6 presents some properties of the adoption strategies of parents.

Lemma 6 (**Adoption Strategies of Parents**). *Assume parents' payoffs satisfy Assumptions 2(a)-(b) ,(d)-(e). Then, the adoption strategies of parents exhibit the following properties.*

(i) *for each (x, q), if $b^p(x, q, a) > 0$ and $\frac{b^p(x,q,a)}{b^p(x,q,f)} > \frac{1-\beta}{1-\beta(1-\delta_x)}$ then $a^p(x, q) = 1$.*

(ii) *for all q, if $\sum_{\mathcal{M}(x_2,q)} f(s) \geq \sum_{\mathcal{M}(x_1,q)} f(s)$ then the best-response of parents satisfies the following: if $a^p(x_1, q) = 1$ then $a^p(x_2, q) = 1$.*

(iii) *for all x, if $\sum_{\mathcal{M}(x,q')} f(s) \geq \sum_{\mathcal{M}(x_1,q)} f(s)$ and $b^p(x, q', a) > 0$ whenever $q' < q$ then the best-response of parents satisfies the following: if $a^p(x, q) = 1$ then $a^p(x, q') = 1$.*

Proof. Assume 2(a). A parent f-matched to child x when the quality is q announces adoption if and only if the following inequalities hold:

$$\frac{b^p(x,q,a)}{1-\beta} > 0 \tag{D.2}$$

$$\frac{b^p(x,q,a)}{1-\beta} > \Big(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\Big)\cdot\frac{b^p(x,q,f)}{1-\beta(1-\delta_x)\big(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\big)} \tag{D.3}$$

(i) Fix (x,q). Assume $b^p(x,q,a)$ is positive then $a^p(x,q)=1$ if and only if inequality D.3 holds. The right-hand side of this inequality is decreasing in $\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)$. Thus, for $a^p(x,q)$ to take value one independent of the endogenous objects $\pi^c(\theta)$ and $\mathcal{M}(x,q)$, the following inequality must hold:

$$\frac{b^p(x,q,a)}{b(x,q)} > \frac{1-\beta}{1-\beta(1-\delta_x)}$$

Or, equivalently $\delta_x > \frac{b^p(x,q,f)-b^p(x,q,a)}{b^p(x,q,a)}\,\frac{1-\beta}{\beta}$.

(ii) Consider a parent f-matched to child x_1 when the quality is q. Assume $a^p(x_1,q)=1$, then inequalities D.2 and D.3 hold. By Assumption 2(b), it follows that $b^p(x_2,q,a)>0$. Hence, $a^p(x_1,q)=1$ implies $a^p(x_2,q)=1$ if the following inequalities holds:

$$\frac{b^p(x_2,q,a)}{b^p(x_2,q,f)} > \frac{b^p(x_1,q,a)}{b^p(x_1,q,f)}$$

and

$$\frac{(1-\beta)\big(1-\pi^c(\theta)\sum_{\mathcal{M}(x_1,q)} f(s)\big)}{1-\beta(1-\delta_{x_1})\big(1-\pi^c(\theta)\sum_{\mathcal{M}(x_1,q)} f(s)\big)} \geq \frac{(1-\beta)\big(1-\pi^c(\theta)\sum_{\mathcal{M}(x_2,q)} f(s)\big)}{1-\beta(1-\delta_{x_2})\big(1-\pi^c(\theta)\sum_{\mathcal{M}(x_2,q)} f(s)\big)}$$

By Assumption 2(d), the first inequality holds. Moreover, since $\delta_{x_2}\geq\delta_{x_1}$ and $\sum_{\mathcal{M}(x_2,q)} f(s)\geq\sum_{\mathcal{M}(x_1,q)} f(s)$, the second inequality holds.

(iii) Consider a parent f-matched to a child x when the quality is q. Assume $a^p(x,q)=1$, then inequalities D.2 and D.3 hold. Also, consider a parent f-matched to child x when the quality is q' such that $q'<q$. Since $b^p(x,q',f)\geq 0$, then $a^p(x,q)=1$ implies $a^p(x,q')=1$ if the following inequalities holds:

$$\frac{b^p(x,q',a)}{b^p(x,q',f)} > \frac{b^p(x,q,a)}{b^p(x,q,f)}$$

$$\frac{(1-\beta)\left(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\right)}{1-\beta(1-\delta_x)\left(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\right)} \geq \frac{(1-\beta)\left(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q')} f(s)\right)}{1-\beta(1-\delta_x)\left(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q')} f(s)\right)}$$

By Assumption 2(e), the first inequality always holds. The second inequality holds since $\sum_{\mathcal{M}(x,q')} f(s) \geq \sum_{\mathcal{M}(x,q)} f(s)$ by assumption.

□

The next lemma presents some properties of the adoption strategies of children.

Lemma 7 (**Adoption Strategies of Children**). *Assume children' payoffs satisfy Assumptions 1(a)-(g), and 5(a)-(c). Moreover, suppose the following*

(*a*) $d(x, q_2) = 0$ *for all* x,

(*b*) $\mathcal{M}(x, q_1)$ *is non-empty for all* x,

(*c*) $\mathcal{M}(x, q_2)$ *is empty for all* x,

(*d*) $\mathcal{M}(x_1, q_1) \subseteq \mathcal{M}(x_2, q_1)$,

(*e*) $s_1 \in \mathcal{M}(x, q_1)$ *implies* $s_2 \in \mathcal{M}(x, q_1)$ *for all* x, *and*

(*f*) $a^p(x_2, q) \geq a^p(x_1, q)$ *for all* q.

Then, the adoption strategies of children are $a^c(x_2, q) \geq a^c(x_1, q)$ *for all* q. *Moreover,* $1 = a^c(x, q_2) \geq a^c(x, q_1)$ *for all* x.

Proof. Fix child (x, q). Since $d^c(x, q) = 0$ by Assumption 1(a) and Lemma 1, she announce adoption if and only if:

$$\frac{b^c(x,q,a)}{1-\beta} > \frac{\left(b^c(x,q,f)+\beta\delta_x\frac{b^c(x,q_2,a)}{1-\beta}\right)\left(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\right)+\pi^c(\theta)\sum_{\mathcal{M}(x,q)} \mathbb{E}_s[\mathcal{C}(x,q)]f(s)}{1-\beta(1-\delta_x)\left(1-\pi^c(\theta)\sum_{\mathcal{M}(x,q)} f(s)\right)} \tag{D.4}$$

Since $\mathcal{M}(x, q_2) = \{\emptyset\}$, inequality D.4 is equal to:

$$\frac{b^c(x,q_2,a)}{1-\beta} > \frac{b^c(x,q_2,f)+\beta\delta_x\frac{b^c(x,q_2,a)}{1-\beta}}{1-\beta\,(1-\delta_x)}$$

By Assumption 1(a), this inequality holds. Hence, $a^c(x, q_2) = 1$ for all x.

For all x, assume that $\mathcal{M}(x, q_1)$ is non-empty , $\mathcal{M}(x_1, q_1) \subseteq \mathcal{M}(x_2, q_1)$, and $s_1 \in \mathcal{M}(x, q_1)$ implies $s_2 \in \mathcal{M}(x, q_1)$. Thus, there are three outcomes:

Case 1: Suppose $\mathcal{M}(x_1, q_1) = \{s_1, s_2\}$ and $\mathcal{M}(x_2, q_1) = \{s_1, s_2\}$. Fixing (x, q_1), inequality D.4 is equal to:

$$b^c(x, q_1, a)\Big\{\big(1-\pi^c(\theta)\big)(1-\beta)+\beta\delta_x\big(1-\pi^c(\theta)\big)+\pi^c(\theta)\Big(g(q_2|s_1)f(s_1)+g(q_2|s_2)f(s_2)\Big)\Big\} >$$
$$\Big(b^c(x, q_1, f) + \beta\delta_x \frac{b^c(x, q_2, a)}{1-\beta}\Big)\cdot$$
$$\big(1-\pi^c(\theta)\big)(1-\beta) + \mathcal{C}(x, q_2)\,\pi^c(\theta)\Big(g(q_2|s_1)f(s_1) + g(q_2|s_2\Big)f(s_2)\big)(1-\beta) \quad \text{(D.5)}$$

Since $d(x, q_2) = 0$ and given the strategies of parents, the value function $\mathcal{C}(x, q_2)$ can take two values:

$$\mathcal{C}(x, q_2) = b^c(x, q_2, f) + \beta\frac{b^c(x,q_2,a)}{1-\beta} \qquad \text{or} \qquad \mathcal{C}(x, q_2) = \frac{b^c(x,q_2,f)+\beta\delta_x\frac{b^c(x,q_2,a)}{1-\beta}}{1-\beta(1-\delta_x)}.$$

Now, since $a^p(x_2, q) \geq a^p(x_1, q)$ for all q, I analyze the following sub-cases:

(1a) Suppose $a^p(x_1, q_2) = 1$ and $a^p(x_2, q_2) = 1$. The child announces adoption if and only if the following inequality holds:

$$\Big\{b^c(x, q_1, a) - b^c(x, q_1, f)\Big\}\big(1-\pi^c(\theta)\big)(1-\beta) >$$
$$\Big\{b^c(x, q_2, a) - b^c(x, q_1, a)\Big\}\beta\delta_x\big(1-\pi^c(\theta)\big)+$$
$$\Big\{b^c(x, q_2, f)(1-\beta)+b^c(x, q_2, a)\beta-b^c(x, q_1, a)\Big\}\big(g(q_2|s_1)f(s_1)+g(q_2|s_2)f(s_2)\big)\pi^c(\theta) \quad \text{(D.6)}$$

where:

$$b^c(x, q_1, a) - b^c(x, q_1, f) > 0 \text{ by Assumption 1(a)}$$
$$b^c(x, q_2, a) - b^c(x, q_1, a) > 0 \text{ by Assumption 1(c)}$$
$$b^c(x, q_2, f)(1-\beta) + b^c(x, q_2, a)\beta - b^c(x, q_1, a) > 0 \text{ by Assumptions 1(c)-(d)}$$

Now, I show that if Equation D.6 holds for child x_1 then it also holds for child x_2. By Assumption 1(e), the following inequality holds:

$$\Big\{b^c(x_2, q_1, a) - b^c(x_2, q_1, f)\Big\}\big(1-\pi^c(\theta)\big)(1-\beta) \geq$$
$$\Big\{b^c(x_1, q_1, a) - b^c(x_1, q_1, f)\Big\}\big(1-\pi^c(\theta)\big)(1-\beta) \quad \text{(D.7)}$$

By Assumptions 1(f) and 5(a), the following inequality holds:

$$\Big\{b^c(x_1,q_2,a)-b^c(x_1,q_1,a)\Big\}\beta\delta_{x_1}\big(1-\pi^c(\theta)\big)\geq \Big\{b^c(x_2,q_2,a)-b^c(x_2,q_1,a)\Big\}\beta\delta_{x_2}\big(1-\pi^c(\theta)\big) \quad \text{(D.8)}$$

By Assumptions 1(f)-(g), the following inequality holds:

$$\Big\{b^c(x_1,q_2,f)(1-\beta)+b^c(x_1,q_2,a)\beta-b^c(x_1,q_1,a)\Big\}\cdot \Big(g(q_2|s_1)f(s_1)+g(q_2|s_2)f(s_2)\Big)\pi^c(\theta)\geq \Big\{b^c(x_2,q_2,f)(1-\beta)+b^c(x_2,q_2,a)\beta-b^c(x_2,q_1,a)\Big\}\cdot \Big(g(q_2|s_1)f(s_1)+g(q_2|s_2)f(s_2)\Big)\pi^c(\theta) \quad \text{(D.9)}$$

Hence, if $a^c(x_1,q_1)=1$ then $a^c(x_2,q_1)=1$.

(1b) Suppose $a^p(x_1,q_2)=0$ and $a^p(x_2,q_2)=0$. The child announces adoption if and only if the following inequality holds:

$$\Big\{b^c(x,q_1,a)-b^c(x,q_1,f)\Big\}\big(1-\pi^c(\theta)\big)(1-\beta)> \Big\{b^c(x,q_2,a)-b^c(x,q_1,a)\Big\}\beta\delta_x\big(1-\pi^c(\theta)\big)+ \Big\{\frac{b^c(x,q_2,f)(1-\beta)}{1-\beta(1-\delta_x)}+\frac{b^c(x,q_2,a)\beta\;\delta x}{1-\beta(1-\delta_x)}-b^c(x,q_1,a)\Big\}\cdot \Big(g(q_2|s_1)f(s_1)+g(q_2|s_2)f(s_2)\Big)\pi^c(\theta) \quad \text{(D.10)}$$

Now, I show that if Equation D.10 holds for child x_1 then it also holds for child x_2. Since Equations D.7 and D.8 hold, then I check whether the following inequality is satisfied:

$$\Big[b^c(x_1,q_2,f)(1-\beta)+b^c(x_1,q_2,a)\beta-b^c(x_1,q_1,a) -\{b^c(x_1,q_2,f)-b^c(x_1,q_1,f)\}\beta(1-\delta_{x_1})\Big]\big(1-\beta+\beta\delta_{x_2}\big)\geq \Big[b^c(x_2,q_2,f)(1-\beta)+b^c(x_2,q_2,a)\beta-b^c(x_2,q_1,a) -\{b^c(x_2,q_2,f)-b^c(x_2,q_1,f)\}\beta(1-\delta_{x_2})\Big]\big(1-\beta+\beta\delta_{x_1}\big) \quad \text{(D.11)}$$

After some algebra, this inequality holds given Assumptions 1(f)-(g) and 5(a). Hence, if $a^c(x_1,q_1)=1$ then $a^c(x_2,q_1)=1$.

(1c) Suppose $a^p(x_1,q_2)=0$ and $a^p(x_2,q_2)=1$. I show that if Equation D.10 holds for child x_1 then Equation D.6 holds for child x_2. Since Equations D.7 and

D.8 hold, I check whether the following inequality is satisfied:

$$\begin{aligned}b^c(x_1,q_2,f)(1-\beta)+b^c(x_1,q_2,a)\beta-b^c(x_1,q_1,a)\\-\beta(1-\delta_{x_1})\Big[b^c(x_1,q_2,a)-b^c(x_1,q_1,a)\Big]\geq\\b^c(x_2,q_2,f)(1-\beta)+b^c(x_2,q_2,a)\beta-b^c(x_2,q_1,a)\\-\beta(1-\delta_{x_1})\Big[b^c(x_2,q_2,f)(1-\beta)+b^c(x_2,q_2,a)\beta-b^c(x_2,q_1,a)\Big]\end{aligned}\tag{D.12}$$

After some algebra, this inequality holds given assumptions 1(b),(f)-(g) and 5(b). Hence, if $a^c(x_1,q_1)=1$ then $a^c(x_2,q_1)=1$.

Case 2: Suppose $\mathcal{M}(x_1,q_1)=\{s_2\}$ and $\mathcal{M}(x_2,q_1)=\{s_2\}$. Fixing (x,q_1), inequality D.4 is equal to:

$$\begin{aligned}b^c(x,q_1,a)\Big\{\big(1-\pi^c(\theta)f(s_2)\big)(1-\beta)+\beta\delta_x\big(1-\pi^c(\theta)f(s_2)\big)+\pi^c(\theta)g(q_2|s_2)f(s_2)\Big\}\\>\\\Big(b^c(x,q_1,f)+\beta\delta_x\frac{b^c(x,q_2,a)}{1-\beta}\Big)\big(1-\pi^c(\theta)f(s_2)\big)(1-\beta)+\mathcal{C}(x,q_2)\,\pi^c(\theta)g(q_2|s_2)f(s_2)(1-\beta)\end{aligned}\tag{D.13}$$

As in the previous case, I analyze the following sub-cases:

(2a) Suppose $a^p(x_1,q_2)=1$ and $a^p(x_2,q_2)=1$. The child announces adoption if and only if the following inequality holds:

$$\begin{aligned}\Big\{b^c(x,q_1,a)-b^c(x,q_1,f)\Big\}\big(1-\pi^c(\theta)f(s_2)\big)(1-\beta)>\\\Big\{b^c(x,q_2,a)-b^c(x,q_1,a)\Big\}\beta\delta_x\big(1-\pi^c(\theta)f(s_2)\big)+\\\Big\{b^c(x,q_2,f)(1-\beta)+b^c(x,q_2,a)\beta-b^c(x,q_1,a)\Big\}g(q_2|s_2)f(s_2)\pi^c(\theta)\end{aligned}\tag{D.14}$$

By Equations D.7, D.8 and D.9, it follows that if Equation D.14 holds for child x_1 then it also holds for child x_2. Hence, if $a^c(x_1,q_1)=1$ then $a^c(x_2,q_1)=1$.

(2b) Suppose $a^p(x_1,q_2)=0$ and $a^p(x_2,q_2)=0$. The child announces adoption if and only if the following inequality holds:

$$\begin{aligned}\Big\{b^c(x,q_1,a)-b^c(x,q_1,f)\Big\}\big(1-\pi^c(\theta)f(s_2)\big)(1-\beta)>\\\Big\{b^c(x,q_2,a)-b^c(x,q_1,a)\Big\}\beta\delta_x\big(1-\pi^c(\theta)f(s_2)\big)+\\\Big\{\frac{b^c(x,q_2,f)(1-\beta)}{1-\beta(1-\delta_x)}+\frac{b^c(x,q_2,a)\beta\;\delta x}{1-\beta(1-\delta_x)}-b^c(x,q_1,a)\Big\}g(q_2|s_2)f(s_2))\pi^c(\theta)\end{aligned}\tag{D.15}$$

By Equations D.7, D.8 and D.11, it follows that if Equation D.15 holds for child x_1 then it also holds for child x_2. Hence, if $a^c(x_1, q_1) = 1$ then $a^c(x_2, q_1) = 1$.

(2c) Suppose $a^p(x_1, q_2) = 0$ and $a^p(x_2, q_2) = 1$. By Equations D.7, D.8 and D.12, it follows that if Equation D.15 holds for child x_1 then equation D.14 holds for child x_2. Hence, if $a^c(x_1, q_1) = 1$ then $a^c(x_2, q_1) = 1$.

Case 3: Suppose $\mathcal{M}(x_1, q_1) = \{s_2\}$ and $\mathcal{M}(x_2, q_1) = \{s_1, s_2\}$.

(3a) Suppose $a^p(x_1, q_2) = 1$ and $a^p(x_2, q_2) = 1$. I show that if Equation D.14 holds for child x_1 then Equation D.6 holds for child x_2. After some algebra, since Equations D.7, D.8 and D.9 hold, it suffices to check whether the following inequality holds:

$$\Big\{b^c_a(x_1, q_2, a) - b^c(x_1, q_1, a)\Big\}\,\beta\delta_{x_1} \geq \Big\{b^c(x_1, q_2, a) - b^c(x_1, q_1, f)\Big\}(1-\beta)+ \Big\{b^c(x_2, q_2, f)(1-\beta) + b^c(x_2, q_2, a)\beta - b^c(x_2, q_1, a)\Big\}g(q_2|s_1) \quad \text{(D.16)}$$

This inequality is satisfied by Assumption 5(c). Hence, if $a^c(x_1, q_1) = 1$ then $a^c(x_2, q_1) = 1$.

(3b) Suppose $a^p(x_1, q_2) = 0$ and $a^p(x_2, q_2) = 0$. I show that if Equation D.15 holds for child x_1 then Equation D.10 holds for child x_2. After some algebra, since Equations D.7, D.8 and D.9 hold, it suffices to check whether the following inequality holds:

$$\Big\{b^c(x_1, q_2, a) - b^c(x_1, q_1, a)\Big\}\,\beta\delta_{x_1} \geq \Big\{b^c(x_1, q_2, a) - b^c(x_1, q_1, f)\Big\}(1-\beta)+ \Big\{\frac{b^c(x_2, q_2, f)(1-\beta)}{1-\beta(1-\delta_{x_2})} + \frac{b^c(x_2, q_2, a)\beta\delta_{x_2}}{1-\beta(1-\delta_{x_2})} - b^c(x_2, q_1, a)\Big\}g(q_2|s_1) \quad \text{(D.17)}$$

This inequality is satisfied by Assumption 5(c). Hence, if $a^c(x_1, q_1) = 1$ then $a^c(x_2, q_1) = 1$.

(3c) Suppose $a^p(x_1, q_2) = 0$ and $a^p(x_2, q_2) = 1$. Since Equations D.7, D.8, D.12 and D.16 hold, it follows that if Equation D.15 holds for child x_1 then Equation D.6 holds for child x_2. Hence, if $a^c(x_1, q_1) = 1$ then $a^c(x_2, q_1) = 1$.

□

Now, I establish Proposition 4.

(i) Fix q. By definition, $a(x, q) = 1$ if and only if $a^c(x, q) = 1$ and $a^p(x, q) = 1$. Since $a^c(x_2, q) \geq a^c(x_1, q)$ and $a^p(x_2, q) \geq a^p(x_1, q)$, then $a(x_2, q) \geq a(x_1, q)$.

(ii) Fix x. Suppose $b^p(x, q_1, a) > 0$, then $b^p(x, q_2, a) > 0$. Since $\frac{b^p(x,q_2,a)}{b^p(x,q_2,f)} \leq \frac{1-\beta}{1-\beta(1-\delta_x)}$ and $\mathcal{M}(x, q_2)$ is empty, then $a^p(x, q_2) = 0$. Thus, $a^p(x, q_1) \geq a^p(x, q_2) = 0$. Since $a^c(x, q_2) \geq a^c(x, q_1)$, it follows that $a(x, q_1) \geq a(x, q_2) = 0$.

E Appendix: Proofs of Empirical Facts and Model Predictions

E.1 Proof of Corollary 1

(i) I show that $A(x_2, q_0) \geq A(x_1, q_0)$, and $A(x_2, q) \geq A(x_1, q)$ for all q. For the first inequality, the result follows from Propositions 2(i) and 4(i), $\mathcal{M}(x_1, q_0) \subseteq \mathcal{M}(x_2, q_0)$ and $a(x_2, q) \geq a(x_1, q)$ for all q. Now, defined the following set $\hat{\mathcal{M}}(x_2, q_0) = \{s \in S | s \in \mathcal{M}(x_2, q_0) \setminus \mathcal{M}(x_1, q_0)\}$, then the following holds:

$$\begin{aligned}
A(x_2, q_0) &= \delta_{x_2} + (1-\delta_{x_2}) \left\{ \pi^c(\theta) \sum_{\mathcal{M}(x_2,q_0)} f(s) \sum_{q'} g(q'|s) \Big[\delta_{x_2} + \big(1-\delta_{x_2}\big) a(x_2, q') \Big] \right\} \\
&= \delta_{x_2} + (1-\delta_{x_2}) \left\{ \pi^c(\theta) \sum_{\mathcal{M}(x_1,q_0)} f(s) \sum_{q'} g(q'|s) \Big[\delta_{x_2} + \big(1-\delta_{x_2}\big) a(x_2, q') \Big] \right. \\
&\quad \left. \underbrace{+ \pi^c(\theta) \sum_{\hat{\mathcal{M}}(x_2,q_0)} f(s) \sum_{q'} g(q'|s) \Big[\delta_{x_2} + \big(1-\delta_{x_2}\big) a(x_2, q') \Big]}_{\geq 0} \right\} \\
&\geq \delta_{x_2} + (1-\delta_{x_2}) \left\{ \pi^c(\theta) \sum_{\mathcal{M}(x_1,q_0)} f(s) \sum_{q'} g(q'|s) \underbrace{\Big[\delta_{x_2} + \big(1-\delta_{x_2}\big)\, a(x_2, q') \Big]}_{\geq \delta_{x_1} + (1-\delta_{x_1})\, a(x_1,q')} \right\} \\
&\geq \underbrace{\delta_{x_1} + (1-\delta_{x_1}) \left\{ \pi^c(\theta) \sum_{\mathcal{M}(x_1,q_0)} f(s) \sum_{q'} g(q'|s) \Big[\delta_{x_1} + \big(1-\delta_{x_1}\big) a(x_1, q') \Big] \right\}}_{= A(x_1,q_0)}
\end{aligned}$$

Now, for the second inequality, fix quality q. By Propositions 1(i) and 4(i), there are three cases to analyze:

<u>Case 1:</u> Suppose $a(x_1, q) = 1$, then $a(x_2, q) = 1$. Thus, $A(x_2, q) = A(x_1, q)$.

<u>Case 2:</u> Suppose $d(x_1, q) = 1$, then:

(2a) If $a(x_2, q) = 1$, then $A(x_2, q) = 1$ and $A(x_1, q) = A(x_1, q_0) \leq 1$.

(2b) If $d(x_2, q) = 1$, then $A(x_2, q) = A(x_2, q_0)$ and $A(x_1, q) = A(x_1, q_0)$.

(2c) If $a(x_2, q) = d(x_2, q) = 0$, then:

$$\begin{aligned}
&A(x_2, q) - A(x_1, q) = \\
&\quad \delta_{x_2} + (1-\delta_{x_2})\, \pi^c(\theta) \sum_{\mathcal{M}(x_2,q)} f(s) \sum_{q'} g(q'|s) \Big[\delta_{x_2} + \big(1-\delta_{x_2}\big) a(x_2, q') \Big] \\
&\qquad - \pi^c(\theta) \sum_{\mathcal{M}(x_1,q_0)} f(s) \sum_{q'} g(q'|s) \Big[\delta_{x_1} + \big(1-\delta_{x_1}\big) a(x_1, q') \Big]
\end{aligned}$$

Note that, it suffices to show that $\delta_{x_2} > \delta_{x_1} + (1-\delta_{x_1})\,\pi^c(\theta)$ holds. Since $\frac{\delta_{x_2}-\delta_{x_1}}{(1-\delta_{x_1})} > \overline{\pi}$, then $A(x_2, q) \geq A(x_1, q)$.

<u>Case 3:</u> Suppose $a(x_1, q) = 0$ and $d(x_1, q) = 0$.

(3a) If $a(x_2, q) = 1$, then $A(x_2, q) = 1$ and $A(x_1, q) \leq 1$.

(3b) If $a(x_2, q) = d(x_2, q) = 0$, then:

$$A(x_2, q) - A(x_1, q) =$$
$$\delta_{x_2} + (1-\delta_{x_2})\,\pi^c(\theta) \sum_{\mathcal{M}(x_2,q)} f(s) \sum_{q'} g(q'|s) \Big[\delta_{x_2} + \big(1-\delta_{x_2}\big)a(x_2, q')\Big]$$
$$-\Big\{\delta_{x_1} + (1-\delta_{x_1})\,\pi^c(\theta) \sum_{\mathcal{M}(x_1,q)} f(s) \sum_{q'} g(q'|s) \Big[\delta_{x_1} + \big(1-\delta_{x_1}\big)a(x_1, q')\Big]\Big\}$$

By Proposition 3(i) and 4(i), the following inequality holds:

$$\sum_{\mathcal{M}(x_2,q)} f(s) \sum_{q} g(q|s)\;\; a(x_2, q) \geq \sum_{\mathcal{M}(x_1,q)} f(s) \sum_{q} g(q|s)\;\; a(x_1, q)$$

Hence, $A(x_2, q) \geq A(x_1, q)$.

(ii) For each child x, I show that $A(x, q_1) \geq A(x, q_2)$. By Propositions 3(ii) and 4(ii), it follows that $A(x, q_2) = \delta_x$. For quality q_1, by Proposition 1(ii), there are three cases to analyze:

<u>Case 1:</u> Suppose $a(x, q_1) = 1$, then $A(x, q_1) = 1$. Thus, $A(x, q_1) \geq A(x, q_2)$.

<u>Case 2:</u> Suppose $d(x, q_1) = 1$, then $A(x, q_1) = \delta_x + (1-\delta_x)\;A(x, q_0)$. Thus, $A(x, q_1) \geq A(x, q_2)$.

<u>Case 3:</u> Suppose $a(x, q_1) = d(x, q_1) = 0$, then:

$$A(x, q_1) = \delta_x + (1-\delta_x)\,\pi^c(\theta) \sum_{\mathcal{M}(x,q_1)} f(s) \sum_{q'} g(q'|s) \Big[\delta_{x_1} + \big(1-\delta_{x_1}\big)a(x_1, q')\Big]$$

Thus, $A(x, q_1) \geq A(x, q_2)$.

E.2 Proof of Corollary 2

(i) I show that $D(x_1, q) \geq D(x_2, q)$ for all q. Fixing quality q, by Propositions 1(i) and 4(i), there are three cases to analyze:

<u>Case 1:</u> Suppose $a(x_1, q) = a(x_2, q) = 1$, then $D(x_1, q) = D(x_2, q)$

Case 2: Suppose $d(x_1, q) = 1$, then:

$$D(x_1,q) - D(x_2,q) =$$
$$(1-\delta_{x_1}) - (1-\delta_{x_2})(1-a(x_2,q))\Big[d(x_2,q) + \big(1-d(x_2,q)\big)\,\pi^c(\theta)\sum_{\mathcal{M}(x_2,q)} f(s)\Big]$$

It follows that $D(x_1, q) \geq D(x_2, q) \geq 0$ holds from $\delta_{x_2} \geq \delta_{x_1}$ and:

$$1 \geq (1-a(x_2,q))\Big[d(x_2,q) + \big(1-d(x_2,q)\big)\,\pi^c(\theta)\sum_{\mathcal{M}(x_2,q)} f(s)\Big] \geq 0$$

Case 3: Suppose $d(x_1, q) = a(x_1, q) = 0$, then:

$$D(x_1,q) - D(x_2,q) = (1-\delta_{x_1})\pi^c(\theta)\sum_{\mathcal{M}(x_1,q)} f(s)$$
$$- (1-\delta_{x_2})(1-a(x_2,q))\Big[d(x_2,q) + \big(1-d(x_2,q)\big)\,\pi^c(\theta)\sum_{\mathcal{M}(x_2,q)} f(s)\Big]$$

(3a) If $a(x_2, q) = 1$, then:

$$D(x_1,q) - D(x_2,q) = (1-\delta_{x_1})\pi^c(\theta)\sum_{\mathcal{M}(x_1,q)} f(s) \geq 0 \tag{E.1}$$

(3b) Suppose $a(x_2, q) = d(x_2, q) = 0$ then:

$$D(x_1,q) - D(x_2,q) = (1-\delta_{x_1})\pi^c(\theta)\sum_{\mathcal{M}(x_1,q)} f(s) - (1-\delta_{x_2})\pi^c(\theta)\sum_{\mathcal{M}(x_2,q)} f(s)$$

For match quality q_2, from Proposition 3 we know that $\mathcal{M}(x, q_2) = \{\emptyset\}$ for all x. Hence, $D(x_1, q_2) \geq D(x_2, q_2)$.
For match quality q_1, since $1 \geq \sum_{\mathcal{M}(x_2,q_1)} f(s)$, it suffices to check that the following inequality holds:

$$(1-\delta_{x_1})\pi^c(\theta)\sum_{\mathcal{M}(x_1,q_1)} f(s) - (1-\delta_{x_2})\pi^c(\theta) \geq 0$$

Proposition 3 shows that $\mathcal{M}(x_1, q_1) = \{s_1, s_2\}$ or $\mathcal{M}(x_1, q_1) = \{s_2\}$. In the first case, $D(x_1, q_1) - D(x_2, q_1) = (1 - \delta_{x_1}) - (1 - \delta_{x_2}) \geq 0$. In the second case, $D(x_1, q_1) - D(x_2, q_1) = (1 - \delta_{x_1})\pi^c(\theta) f(s_2) - (1 - \delta_{x_2})\pi^c(\theta)$ which is positive if and only if $f(s_2) \geq \frac{(1-\delta_{x_2})}{(1-\delta_{x_1})}$.

(ii) Fixing child x, suppose that $a(x, q_1) = a(x, q_2) = 0$. From Propositions 1(ii) and 3(ii), $d(x, q_1) \geq d(x, q_2)$ and $\sum_{\mathcal{M}(x,q_1)} f(s) \geq \sum_{\mathcal{M}(x,q_2)} f(s) = 0$ respectively. Hence, the following inequality holds:

$$\begin{aligned}
D(x, q_1) &= (1-\delta_x)\Big[d(x, q_1) + \big(1 - d(x, q_1)\big)\, \pi^c(\theta) \sum_{\mathcal{M}(x,q_1)} f(s)\Big] \\
&\geq (1-\delta_x)\Big[d(x, q_2) + \big(1 - d(x, q_2)\big)\, \pi^c(\theta) \sum_{\mathcal{M}(x,q_1)} f(s)\Big] \\
&\geq (1-\delta_x)\Big[d(x, q_2) + \big(1 - d(x, q_2)\big)\, \pi^c(\theta) \sum_{\mathcal{M}(x,q_2)} f(s)\Big] = D(x, q_2)
\end{aligned}$$

E.3 Proof of Corollary 3

The result follows from Propositions 1(i) and 2(i): $d(x_1, q) \geq d(x_2, q)$ for all q, and $\mathcal{M}(x_1, q_0) \subseteq \mathcal{M}(x_2, q_0)$. Let $\hat{\mathcal{M}}(x_2, q_0) = \{s \in S | s \in \mathcal{M}(x_2, q_0) \setminus \mathcal{M}(x_1, q_0)\}$, then the following inequality holds:

$$\begin{aligned}
M(x_2) = \; & \pi^c(\theta) \sum_{\mathcal{M}(x_2,q_0)} f(s) \sum_q g(q|s)\, \big(1 - d(x_2, q)\big) \\
\geq \; & \pi^c(\theta) \sum_{\mathcal{M}(x_2,q_0)} f(s) \sum_q g(q|s)\, \big(1 - d(x_1, q)\big) \\
\geq \; & \pi^c(\theta) \Big[\sum_{\hat{\mathcal{M}}(x_2,q_0)} f(s) \sum_q g(q|s)\, \big(1 - d(x_1, q)\big) \\
& + \sum_{\mathcal{M}(x_1,q_0)} f(s) \sum_q g(q|s)\, \big(1 - d(x_1, q)\big)\Big] \\
\geq \; & \pi^c(\theta) \sum_{\mathcal{M}(x_1,q_0)} f(s) \sum_q g(q|s)\, \big(1 - d(x_1, q)\big) = M(x_1)
\end{aligned}$$

Hence, $M(x_2) \geq M(x_1)$.

E.4 Proof of Corollary 4

(i) Fixing quality q, I show that $U(x_1, q) \geq U(x_2, q)$ for all q. From Propositions 1(i) and 4(i), there are three cases to analyze:

<u>Case 1:</u> Suppose $a(x_1, q) = a(x_2, q) = 1$ then $U(x_1, q) = U(x_2, q)$

Case 2: Suppose $d(x_1, q) = 1$ then:

$$\begin{aligned} U(x_1,q) - U(x_2,q) = (1-\delta_{x_1}) \Big(1 - M(x_1)\Big) \\ - (1-\delta_{x_2}) \left(1 - a(x_2,q)\right) \Big\{ d(x_2,q) \Big(1 - M(x_2)\Big) \\ + \left(1 - d(x_2,q)\right) \pi^c(\theta) \sum_{\mathcal{M}(x_2,q)} f(s) \sum_{q'} g(q'|s)\, d(x_2,q') \Big\} \end{aligned}$$

(2a) If $a(x_2, q) = 1$, then $U(x_1, q) - U(x_2, q) = (1 - \delta_{x_1})(1 - M^3(x_1)) \geq 0$.

(2b) If $d(x_2, q) = 1$, then:

$$U(x_1,q) - U(x_2,q) = (1-\delta_{x_1})(1 - M(x_1)) - (1-\delta_{x_2})(1 - \gamma^3(x_2))$$

By Corollary 3, it follows that $U(x_1, q) \geq U(x_2, q)$.

(2c) If $a(x_2, q) = d(x_2, q) = 0$, then:

$$\begin{aligned} &U(x_1,q) - U(x_2,q) = \\ &(1-\delta_{x_1})(1 - M(x_1)) - (1-\delta_{x_2})\pi^c(\theta) \sum_{\mathcal{M}(x_2,q)} f(s) \sum_{q'} g(q'|s)\, d(x_2,q') \end{aligned}$$

For match quality q_2, Proposition 3 shows that $\mathcal{M}(x, q_2) = \{\emptyset\}$ for all x. Hence, $U(x_1, q_2) - U(x_2, q_2) = (1 - \delta_{x_1})(1 - M(x_1)) \geq 0$.
For match quality q_1, since the following holds:

$(1 - \pi^c(\theta)) \geq (1 - M(x_1))$ and $1 \geq \sum_{\mathcal{M}(x_2,q_1)} f(s) \sum_{q'} g(q'|s)\, d(x_2, q')$

it suffices to check that the following inequality holds:

$$(1-\delta_{x_1})(1 - \pi^c(\theta)) - (1-\delta_{x_2})\pi^c(\theta) \geq 0$$

which holds if and only if $\frac{1-\delta_{x_1}}{2-\delta_{x_1}-\delta_{x_2}} \geq \overline{\pi}$.

Case 3: Suppose $a(x_1, q) = 0$ and $d(x_1, q) = 0$ then:

$$\begin{aligned} U(x_1,q) - U(x_2,q) = (1-\delta_{x_1})\pi^c(\theta) \sum_{\mathcal{M}(x_1,q)} f(s) \sum_{q'} g(q'|s)\, d(x_1,q') \\ - (1-\delta_{x_2}) \left(1 - a(x_2,q)\right) \Big\{ d(x_2,q) \Big(1 - M(x_2)\Big) \\ + \left(1 - d(x_2,q)\right) \pi^c(\theta) \sum_{\mathcal{M}(x_2,q)} f(s) \sum_{q'} g(q'|s)\, d(x_2,q') \Big\} \end{aligned}$$

(3a) If $a(x_2, q) = 1$, then:

$$U(x_1, q) - U(x_2, q) = (1 - \delta_{x_1})\pi^c(\theta) \sum_{\mathcal{M}(x_1,q)} f(s) \sum_{q'} g(q'|s)\, d(x_1, q') \geq 0$$

(3b) If $a(x_2, q) = d(x_2, q) = 0$, then:

$$\begin{aligned} U(x_1, q) - U(x_2, q) = (1 - \delta_{x_1})\pi^c(\theta) \sum_{\mathcal{M}(x_1,q)} f(s) \sum_{q'} g(q'|s)\, d(x_1, q') \\ - (1 - \delta_{x_2})\pi^c(\theta) \sum_{\mathcal{M}(x_2,q)} f(s) \sum_{q'} g(q'|s)\, d(x_2, q') \end{aligned}$$

For match quality q_2, Proposition 3 states that $\mathcal{M}(x, q_2) = \{\emptyset\}$ for all x. Hence, $U(x_1, q) = U(x_2, q)$.
For match quality q_1, since $1 \geq \sum_{\mathcal{M}(x_2,q_1)} f(s)$, it suffices to check that the following inequality holds:

$$(1 - \delta_{x_1})\pi^c(\theta) \sum_{\mathcal{M}(x_1,q_1)} f(s) - (1 - \delta_{x_2})\pi^c(\theta) \geq 0$$

Proposition 3 shows that $\mathcal{M}(x_1, q_1) \neq \{\emptyset\}$ and $\mathcal{M}(x_1, q_1) = \{s_1, s_2\}$ or $\mathcal{M}(x_1, q_1) = \{s_2\}$. In the first case, $D(x_1, q_1) - D(x_2, q_1) = (1 - \delta_{x_1}) - (1 - \delta_{x_2}) \geq 0$. In the second case, $D(x_1, q_1) - D(x_2, q_1) = (1 - \delta_{x_1})\pi^c(\theta) f(s_2) - (1 - \delta_{x_2})\pi^c(\theta)$ which is positive if and only if $f(s_2) \geq \frac{(1-\delta_{x_2})}{(1-\delta_{x_1})}$.

(ii) Fixing child x, suppose that $a(x, q_1) = 0$ and $a(x, q_2) = 0$. By Propositions 1(ii) and 3(ii) it follows that $d(x, q_1) \geq d(x, q_2)$ and $\sum_{\mathcal{M}(x,q_1)} f(s) \geq \sum_{\mathcal{M}(x,q_2)} f(s) = 0$ respectively. Hence, the following inequality holds:

$$\begin{aligned} U(x, q_1) &= (1 - \delta_x)\Big\{ d(x, q_1)\, \big(1 - M(x)\big) \\ &+ \big(1 - d(x, q_1)\big)\, \pi^c(\theta) \sum_{\mathcal{M}(x,q_1)} f(s) \sum_{q'} g(q'|s)\, d(x, q') \Big\} \\ &\geq (1 - \delta_x)\Big\{ d(x, q_1)\, \big(1 - M(x)\big) \\ &+ \big(1 - d(x, q_1)\big)\, \underbrace{\pi^c(\theta) \sum_{\mathcal{M}(x,q_2)} f(s) \sum_{q'} g(q'|s)\, d(x, q')}_{=0} \Big\} \\ &\geq (1 - \delta_x)\, d(x, q_2)\, \big(1 - M(x)\big) = U(x, q_2) \end{aligned}$$